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Complete sets of orthogonal tableaux

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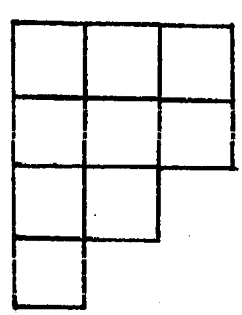
TABLE OF CONTENTS

I. INTRODUCTION

In the early part of this century, Alfred Young developed a recipe for writing down the matrices of the natural irreducible representations of the symmetric group S_n [8, p. 399]. In this dissertation, we begin the construction of bases of orthogonal natural idempotents from tableaux which are not necessarily standard. These bases greatly simplify the calculation of the natural matrices.

We begin with the definition of a tableau. Given a positive integer n, let $m_1 \geq m_2 \geq \ldots \geq m_r$ be positive integers such that $m_1 + m_2 + \ldots + m_r = n$. From this partition of n, construct a frame of r rows of squares. These rows have lengths $m_1, m_2, \ldots, m_r$ and are arranged with their left hand ends directly under one another. (A comma following a subscript can be mistaken for a prime on the subscript. The reader should remember that commas are slanted while primes are vertical.) We denote the frame by $(m) = (m_1, m_2, \ldots, m_r)$. Thus, (3,3,2,1) denotes the frame:



By placing the numbers 1 to n in any order into the n squares, we obtain a tableau. Thus, there are n! tableaux for a given frame. A tableau is called standard if the numbers increase in every row from left to right and in every column downwards. For example, there are five standard tableaux for the frame (3,2):

123	124	125	134	135
45	35	34	25	24

Using the standard tableaux for a given frame, Young was able to construct basis elements with which he could compute the matrices for an irreducible representation of the symmetric group S_n [8].

We call a set of tableaux orthogonal if, given any two tableaux T and T' in the set, we can find two numbers that occur together in some column of T and in some row of T'. For a given frame, we say an orthogonal set of tableaux is complete if its cardinality is the same as that of the set of standard tableaux. If the standard tableaux for a given frame are orthogonal, the construction of the matrices for the natural irreducible representation of S_n is relatively easy [6, p. 51]. In general, this is not the case. It was necessary for Young to go through a tedious and complicated procedure of modifications, called chains [2, p. 119], in order to calculate the matrices. Computer programs which calculate the matrices for the natural representation of S_n have been written using Young's prescription [7], but they tend to be rather long and complicated. For $n>9$, the programs become difficult to run, due to the time and memory required to calculate the chains. This fact prompted Ronald Smith to pose the question: "Why not choose complete sets of orthogonal tableaux to construct the basis elements?" Although Alfred Young considered other definitions for his standard tableaux [8, p. 395], there is no indication in any of his papers that he considered using complete sets of orthogonal tableaux. For $n \leq 4$, the standard tableaux are orthogonal. In [6, p. 51], Rutherford mentions this fact. He notes that for $n>4$, it is unfortunate that the standard tableaux are not necessarily orthogonal, but he does not mention the possibility of choosing complete sets of tableaux which are orthogonal. Thus, we are led to the following problem:

> Determine for which frames complete sets of orthogonal tableaux exist, and when they exist, classify them.

In a correspondence in the fall of 1979, G. de B. Robinson (Young's sole Ph.D. student) stated that he did not know of anybody who has considered this problem.

There are several advantages to a basis derived from a complete set of orthogonal tableaux. The major advantage is that no chains need to be calculated. This makes the actual computation of the matrices relatively easy. A second advantage is that the entries of the matrices will be restricted to +1, -1, and 0, which allows easy storage of the matrices. A third advantage is that given such a basis, one can immediately calculate, by inspection, any entry for the matrix of any permutation of S_n. For $n>4$, this would be impossible to do using the basis derived from the standard tableaux. Since the matrices become huge for $n>7$, such a property is desirable. A fourth advantage is the relative simplicity of the basis elements, a property which is nice when studying the group algebra.

With these advantages as our goal, we begin in this dissertation the search for complete sets of orthogonal tableaux. In Chapter II, we state the results from the literature which we use in our search for these sets. In Chapter III, we prove some general results which we use in Chapter IV, where we begin the actual construction of complete sets of orthogonal tableaux. In Chapter IV, we obtain some results for frames with two rows, and we give a procedure for constructing complete sets of orthogonal tableaux which is effective for some frames other than those with two rows. In the Appendix, we list some examples of complete sets of orthogonal tableaux.

II. PRELIMINARIES

In this chapter, we list the results in the literature that will be used in the following chapters. Throughout this chapter, n will be some fixed integer, and all frames will be assumed to have n squares.

Consider the group algebra $C(S_n)$ of S_n over the complex numbers. By Wedderburn's and Maschke's Theorems, $C(S_n)$ is isomorphic to a direct sum of matrix rings over the complex numbers [3, p. 51]. Using the standard tableaux, Young was able to construct basis elements which reveal this property. By choosing complete sets of orthogonal tableaux, we greatly simplify this construction. We start by giving some definitions.

For a given tableau T, we consider two special types of permutations, the p and the q. We denote by p those permutations which interchange only the numbers in each row amongst themselves. Thus, the rows are fixed as sets, not necessarily elementwise, under the p, and the p are called horizontal permutations for T. Similarly, the q are those permutations which interchange only the numbers in each column amongst themselves, and the q are called vertical permutations for T. Construct $P = \Sigma p$ and $Q = \Sigma \mathrm{sgn}(q) q$, where the first summation extends over all horizontal permutations for T and the second summation extends over all vertical permutations for T ($\mathrm{sgn}(q) = +1$ when q is even and -1 when q is odd). If f denotes the number of standard tableaux for the frame corresponding to T, then we have the following result [2, p. 109]:

<u>Theorem 2.1</u>: In $C(S_n)$, $PQPQ = \kappa PQ$, where $\kappa = n!/f$, and the left ideal of $C(S_n)$ generated by PQ is minimal.

Thus, $e = 1/\kappa\ PQ$ is an idempotent in $C(S_n)$, and we call e the idempotent derived from T. Note that $PQ = \sum_{p}\sum_{q} \mathrm{sgn}(q)pq$. If $p_1q_1 = p_2q_2$, then $p_2^{-1}p_1 = q_2q_1^{-1} = I$, since $p_2{}^{-1}p_1$ leaves the rows of T fixed and $q_2q_1^{-1}$ leaves the columns of T fixed. Therefore, $p_1 = p_2$, $q_1 = q_2$, and we can write $PQ = \sum_{pq} \mathrm{sgn}(q)pq$.

The n! left ideals generated by the idempotents derived from all tableaux which belong to the frame (m) generate a simple two-sided ideal of $C(S_n)$ which is isomorphic to a matrix ring over the complex numbers [2, p. 111]. We shall call this two-sided ideal the ideal corresponding to the frame (m). If we denote the identity for this two-sided ideal by e_0, then we have the following result [6, p. 65]:

<u>Theorem 2.2</u>: Let $T_1, T_2, \ldots, T_{n!}$ be the distinct tableaux for the frame (m). Let $e_1, e_2, \ldots, e_{n!}$ be the corresponding idempotents derived from these tableaux. Then $\sum_{i=1}^{n!} e_i = \kappa e_0$, where $\kappa = n!/f$.

We use the e_i to construct elements in this two-sided ideal which reveal its property of being isomorphic to a matrix ring over the complex numbers.

Given a tableau T and a permutation r, we denote by rT the tableau obtained by applying the permutation r to the numbers of T. For example, if r = (135) and T is $\begin{array}{l}123\\45\end{array}$, then rT is $\begin{array}{l}325\\41\end{array}$. We note that if p is a horizontal permutation for T, then rpr^{-1} is a horizontal permutation for rT. A similar result holds for vertical permutations of T. We have the following result [2, p. 105]:

Theorem 2.3: Let T be a tableau and r be a permutation. If e is the idempotent derived from T, then rer^{-1} is the idempotent derived from rT.

If p is a horizontal permutation for tableau T and q is a vertical permutation for T, then $pq = (pqp^{-1})p$. Thus, we can get from T to pqT by first applying a horizontal permutation to T, and then applying a vertical permutation to pT. This yields [2, p. 106]:

Theorem 2.4: Let T be a tableau and r be a permutation. Then there exists two numbers which occur in the same row in T and in the same column of rT if and only if $r \neq pq$ for every horizontal permutation p of T and every vertical permutation q of T.

Since the set of horizontal permutations for T forms a group, we have $pP = Pp = P$ for any p which is a horizontal permutation for T. Similarly, $qQ = Qq = \mathrm{sgn}(q)Q$ for any q which is a vertical permutation for T. Thus, $pe = e$ and $eq = \mathrm{sgn}(q)e$. Using this, one can show [2, p. 107]:

Theorem 2.5: Let T and T' be tableaux. If there exists two numbers which occur in one row in T and in one column in T', then $Q'P = 0$, and consequently $e'e = 0$.

Theorem 2.6: Let T be a tableau and $r = pq$, with p a horizontal permutation for T and q a vertical permutation for T. If $T' = rT$, then $Q'rP = \mathrm{sgn}(q)Q'P$, and consequently $e'e = \mathrm{sgn}(q)e're = \mathrm{sgn}(q)pqe \neq 0$.

These two theorems allow us to determine how the idempotents from different tableaux multiply.

Let $T_1, T_2, \ldots, T_{n!}$ be the distinct tableaux corresponding to a given frame. For each i, let e_i be the idempotent derived from T_i. For each i and j, let s_{ij} denote the permutation which takes the tableau T_j into the tableau T_i, i.e., $T_i = s_{ij}T_j$. It is clear that we have $s_{ij}^{-1} = s_{ji}$, $s_{ij}s_{j\ell} = s_{i\ell}$, and $e_i = s_{ij}e_js_{ji}$. Let $e_{ij} = s_{ij}e_j$ for each $i,j = 1,2,\ldots,n!$. Note that we also have $e_{ij} = e_is_{ij}$. Given i and j, we know from Theorems 2.4-2.6 that either $e_ie_j = 0$ or $s_{ij} = pq$, where p is a horizontal permutation for T_j and q is a vertical permutation for T_j. In the latter case,

$$e_i e_j = \operatorname{sgn}(q) e_i s_{ij} e_j = \operatorname{sgn}(q) s_{ij} e_j e_j = \operatorname{sgn}(q) s_{ij} e_j.$$

For each i and j, define

$$\varepsilon_{ij} = \begin{cases} 0 \text{ if } e_i e_j = 0 \\ \operatorname{sgn}(q) \text{ if } e_i e_j \neq 0, \text{ where } q \text{ is as above} \end{cases} . \qquad (2.1)$$

Then we have the important result [6, p. 22]:

<u>Theorem 2.7</u>: $e_{ij} e_{k\ell} = \varepsilon_{jk} e_{i\ell}$.

This result has the following generalization [6, p. 20]:

<u>Theorem 2.8</u>: If x is an element of $C(S_n)$, then $e_{ij} x e_{k\ell} = \lambda e_{i\ell}$, where λ/κ is the coefficient of the identity permutation in $e_{kj} x$ ($\kappa = n!/f$, where f is the number of standard tableaux for the frame under consideration).

Next, we consider tableaux from different frames. These frames are ordered as follows: If $(m) = (m_1, m_2, \ldots, m_r)$ and $(m') = (m_1', m_2', \ldots, m_s')$ are different frames, $(m) > (m')$ if the first nonzero difference $m_i - m_i'$ is positive. One can show [6, p. 21]:

<u>Theorem 2.9</u>: If T belongs to the frame (m) and T' belongs to the frame (m'), and (m)>(m'), then there exists two numbers which occur in one row in T and in one column in T'.

<u>Theorem 2.10</u>: If $(m) \neq (m')$, then $e'_{ij}xe_{k\ell} = 0$ for all x in $C(S_n)$.

Given a frame (m), if (m') is the frame obtained by transposing the rows and columns of (m), then we call (m') the frame conjugate to (m). For example, (2,2,1,1,1) is the frame conjugate to (5,2).

Finally, we consider the standard tableaux. One can show the following [2, p. 112]:

<u>Theorem 2.11</u>: The number of standard tableaux which belong to the frame $(m_1, m_2,\ldots,m_r)$ is $f(m_1, m_2,\ldots,m_r) =$ $n! \dfrac{\prod_{i<k} (\ell_i-\ell_k)}{\ell_1!\ell_2!\ldots\ell_r!}$, where $\ell_i = m_i + r - i$, $i = 1,2,\ldots,r$.

If we define $f(m_1,m_2,\ldots,m_r) = 0$ if $m_j<m_{j+1}$ for some j, and $f(m_1,m_2,\ldots,m_r,0) = f(m_1,m_2,\ldots,m_r)$, then we have the following [2, p. 121]:

<u>Theorem 2.12</u>: $f(m_1, m_2,\ldots,m_r) = f(m_1-1, m_2,\ldots,m_r) +$ $f(m_1, m_2-1,\ldots,m_r) +\ldots+ f(m_1, m_2,\ldots,m_r-1)$.

We also have that $f(m_1,m_2,\ldots,m_r)$ is the degree of the representation of S_n corresponding to the frame $(m_1,m_2,\ldots,m_r)$ [2, p. 112]. This implies the following result [2, p. 112]:

Theorem 2.13: If the different frames with n squares are characterized by means of the index j, then $\sum_j f_j^{\,2} = n!$.

Let $T_1,T_2,\ldots,T_f$ be the standard tableaux for a given frame. We say that these standard tableaux are in dictionary order if the following is the case: $i<k$, if, in reading the numbers ρ of T_i and σ of T_k as lines of a book, the first nonzero difference $\sigma-\rho$ is positive. In the example in Chapter I, the five standard tableaux of (3,2) are given in dictionary order. One can show [2, p. 113]:

Theorem 2.14: If T_i and T_k are standard tableaux with $i<k$, then $e_k e_i = 0$.

Theorem 2.15: The ideal corresponding to the frame (m) is the direct sum of the left ideals generated by the idempotents derived from the standard tableaux of (m).

Now for $n\leq 4$ and the standard tableaux in dictionary order, we also have $e_i e_k = 0$ for $i<k$. So by Theorem 2.7, we have $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$ for the standard tableaux if $n\leq 4$, where $\delta_{jk} = 0$ if $j\neq k$, 1 if $j=k$. In this case, we have that

the e_{ij} act like matrix units. These e_{ij} yield an irreducible representation for S_n, which can be calculated using Theorem 2.8 as follows:

To find the i,j entry $[\pi]_{ij}$ of the matrix corresponding to the permutation π, apply π to T_j, i.e., $\pi T_j = T_r$, where T_r is some tableau, not necessarily standard. Then $[\pi]_{ij} = \varepsilon_{ir}$. Thus, the matrix for π has only +1, -1, and 0 as entries.

This prescription for $n \leq 4$ is slightly different than that given in [2], but yields the same matrices. In [2], Boerner applies π^{-1} to T_i rather than applying π to T_j. The reason for doing this is that for $n>4$, the standard tableaux do not necessarily satisfy $e_i e_k = 0$ if $i<k$. It is necessary to modify the e_i that do not satisfy this by replacing them with $e_i' = e_i w_i$, where w_i is an integer combination of permutations [8, p. 361]. Then, rather than having the ε_{rj} as the entries of the matrix corresponding to a permutation, one has sums of products of ε_{rj} (chains) as the entries. Using the e_i', Boerner gives a precise prescription for calculating the matrices for the natural irreducible representation of S_n which corresponds to a given frame [2, p. 119]. This is the same prescription that Young gives [8, p. 399]. In [7], there is a computer program to calculate these matrices using Boerner's prescription. The program is

somewhat long and tedious, the reason being the need to calculate the chains.

By Theorems 2.4-2.7 and definition, a complete set of orthogonal tableaux is a set of tableaux, $\{T_1, T_2, \ldots, T_f\}$ such that $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$. If one had a complete set of orthogonal tableaux, no chains need to be calculated, and the same prescription as that given above for $n \leq 4$ may be used. This would make the calculation of the matrices much simpler.

As a final note, we mention that in [6], Rutherford gives a slightly different and considerably easier method to construct the $e_i' = e_i w_i$. This allows him to give a much shorter proof that $e_{ij}'e_{k\ell}' = \delta_{jk}e_{i\ell}'$. He then shows that his definition of the e_i' is equivalent to that of Young, mentions that the matrix elements of the natural representation are the coefficients of the various permutations in $\kappa e_i' (\kappa = n!/f)$, and states that he will not make much further use of the natural representation. Earlier in the book [6, p. 51], he mentions the difficulty of calculating the matrices for the natural representation; however, nothing more is said about calculating these matrices. It turns out that by using Rutherford's construction for the e_i', a much simpler prescription for calculating the matrices can be given than that given in [2]. This prescription does not appear to be in the literature. Since Boerner [2] references Rutherford's book [6], it appears that he also was unaware of the much simpler prescription that is possible using Rutherford's construction of the e_i'. We will derive this prescription here, denoting the e_i' by g_{ii}.

For a given frame, let $T_1, T_2, \ldots, T_f$ be the standard tableaux in dictionary order. Let $A(I)$ be the fxf matrix with $[A(I)]_{ij} = \varepsilon_{ij}$ for $i,j = 1,2,\ldots,f$. By Theorem 2.14 and (2.1), $\varepsilon_{ij} = 0$ if $i>j$. Also, by Theorem 2.1 and (2.1), $\varepsilon_{ii} = 1$ for $i = 1,2,\ldots,f$. Thus, $A(I)$ has determinant 1, and hence is invertible. Let $w_{ij} = [A(I)^{-1}]_{ij}$ for $i,j = 1,2,\ldots,f$. Then $\sum_{k=1}^{f} w_{jk}\varepsilon_{k\ell} = \delta_{j\ell}$ for $j,\ell = 1,2,\ldots,f$. Define

$g_{ij} = \sum_{k=1}^{f} e_{ik}w_{jk}$. Then by Theorem 2.7, we have:

$$
\begin{aligned}
g_{ij}g_{mn} &= \left(\sum_{k=1}^{f} e_{ik}w_{jk}\right)\left(\sum_{\ell=1}^{f} e_{m\ell}w_{n\ell}\right) \\
&= \sum_{k,\ell=1}^{f} w_{jk}e_{ik}e_{m\ell}w_{n\ell} \\
&= \sum_{k,\ell=1}^{f} w_{jk}\varepsilon_{km}e_{i\ell}w_{n\ell} \\
&= \left(\sum_{k=1}^{f} w_{jk}\varepsilon_{km}\right)\left(\sum_{\ell=1}^{f} e_{i\ell}w_{n\ell}\right) \\
&= \delta_{jm}g_{in}.
\end{aligned}
$$

Thus, the g_{ij} act like matrix units. These g_{ij} are equivalent to the e'_{ij} [6, p. 54].

Let π be a permutation, and consider $e_k \pi e_j$. Let $T_r = \pi T_j$ (T_r is not necessarily standard). Then $T_k = s_{kr} T_r = s_{kr} \pi T_j$, so $s_{kj} = s_{kr}\pi$. Therefore, $e_k \pi e_j = e_k(\pi e_j \pi^{-1})\pi = e_k e_r \pi = \varepsilon_{kr} e_{kr} \pi = \varepsilon_{kr} e_k s_{kr} \pi = \varepsilon_{kr} e_k s_{kj} = \varepsilon_{kr} e_{kj}$. Denote ε_{kr} by ε^{π}_{kj}. Let $A(\pi)$ be the fxf matrix with $[A(\pi)]_{kj} = \varepsilon^{\pi}_{kj}$ for $k,j = 1,2,\ldots,f$. If $[\pi]_{ij}$ denotes the i,j entry in the matrix corresponding to the permutation π, then we have:

$$\begin{aligned}
[\pi]_{ij} g_{ij} = g_{ii} \pi g_{jj} &= \left(\sum_{k=1}^{f} e_{ik} w_{ik}\right) \pi \left(\sum_{\ell=1}^{f} e_{j\ell} w_{j\ell}\right) \\
&= \sum_{k,\ell=1}^{f} w_{ik} s_{ik} (e_k \pi e_j) s_{j\ell} w_{j\ell} \\
&= \sum_{k,\ell=1}^{f} w_{ik} s_{ik} \varepsilon^{\pi}_{kj} e_{kj} s_{j\ell} w_{j\ell} \\
&= \left(\sum_{k=1}^{f} w_{ik} \varepsilon^{\pi}_{kj}\right) \left(\sum_{\ell=1}^{f} e_{i\ell} w_{j\ell}\right) \\
&= \left(\sum_{k=1}^{f} w_{ik} \varepsilon^{\pi}_{kj}\right) g_{ij}.
\end{aligned}$$

Therefore, $[\pi]_{ij} = \sum_{k=1}^{f} w_{ik} \varepsilon^{\pi}_{kj}$.

Thus, $[\pi] = A(I)^{-1} A(\pi)$.

Since $A(I)$ has determinant 1 and only +1, -1, and 0 as entries, $A(I)^{-1}$, and hence $[\pi]$, have only integer entries. This gives us the simpler prescription for calculating the matrices of an irreducible representation of S_n which corresponds to a given frame:

Let $T_1, T_2, \ldots, T_f$ be the standard tableaux in dictionary order. For a permutation π, form the matrix $A(\pi)$ with $[A(\pi)]_{ij} = \varepsilon_{ij}^{\pi} = \varepsilon_{ir}$, where $T_r = \pi T_j$ (see the definition preceding Theorem 2.7). Then $[\pi] = A(I)^{-1}A(\pi)$, where I is the identity permutation in S_n.

We note that this procedure will work for any set of f tableaux provided $A(I)$ is invertible. The entries for the matrices may not be integers, however, if the determinant of $A(I)$ is something other than +1 or -1.

In the above setting, the problem of finding a complete set of orthogonal tableaux for a given frame is equivalent to finding f tableaux such that $A(I)$ is the identity matrix. Thus, we have the following result:

<u>Theorem 2.16</u>: If $\{T_1, T_2, \ldots, T_f\}$ is a complete set of orthogonal tableaux for the frame (m), and e_0 is the identity for the ideal corresponding to (m), then

$$\sum_{i=1}^{f} e_i = e_0.$$

III. GENERAL RESULTS

In this chapter, we state and prove results that are used in finding complete sets of orthogonal tableaux. In Section A, these results apply to all tableaux. In Section B, the results are proved only for flat tableaux (tableaux with exactly two rows).

A. All Tableaux

Ideally one would like to have a criterion to determine whether or not a given set of tableaux can be extended to a complete set of orthogonal tableaux. This would allow a complete set of orthogonal tableaux to be built up inductively from a single tableau. Unfortunately, such a criterion, as yet, does not exist. In this section, we prove some results which give us a start in this direction.

Suppose two of the rows of a tableau T have equal length. Let T' be the tableau obtained from T by interchanging these two rows. Since the sets of horizontal and vertical permutations for T and T' are the same, the idempotent e derived from T is equal to the idempotent e' derived from T'. Similarly, interchanging columns of equal length in a tableau leaves the idempotent unchanged. For example, the following four tableaux all yield the same idempotent:

123	456	213	546
456	123	546	213
78	78	87	87 .

<u>Definition</u>: Tableaux T and T' are said to be <u>equivalent</u> if e = e', where e is the idempotent derived from T and e' is the idempotent derived from T'.

If T' is obtained from T by interchanging rows of T of equal length and/or interchanging columns of T of equal length, then T' and T are equivalent. Conversely, if T' and T are equivalent, then PQ = P'Q'. By the remarks following Theorem 2.1,

$$\sum_{pq} \operatorname{sgn}(q)pq = \sum_{p'q'} \operatorname{sgn}(q')p'q'.$$

Suppose $P \neq P'$. Then there exists c and d such that c and d are together in a row of T, but c and d are not together in a row of T'. Thus, (cd) is a horizontal permutation for T, but not for T'. Therefore, (cd) = p'q' for some horizontal permutation p' for T' and some vertical permutation q' for T'. By Theorem 2.4 and the remarks preceding it, we can go from T' to (cd)T' by first applying a horizontal permutation to T' and then applying a vertical permutation to the resulting tableau. Since c and d are in different rows of T', the horizontal permutation we must apply to T' is the identity. Thus, (cd) = q', a vertical permutation for T'. Since sgn(q') = -1, (cd) occurs in PQ with a sign of +1 and in P'Q' with a sign of -1. Since the group elements of S_n are linearly independent in $C(S_n)$, this is a contradiction. Therefore, P = P'. Similarly, Q = Q'. Thus, T and T' have the same group of horizontal permutations and the same group of vertical permutations. This implies that one can go from T to T' by interchanging rows of T of equal length and/or interchanging columns of T of equal length. In searching for a complete set of orthogonal tableaux, we need only consider one T from each of the equivalence classes:

$$[T] = \{T' : e=e'\}.$$

For a given frame, suppose $\{T_1, T_2, \ldots, T_f\}$ is a complete set of orthogonal tableaux. Then $e_i e_j = \delta_{ij} e_j$ for $i,j = 1,2,\ldots,f$. If π is a permutation, then

$$(\pi e_i \pi^{-1})(\pi e_j \pi^{-1}) = \delta_{ij}(\pi e_j \pi^{-1}) \text{ for } i,j = 1,2,\ldots,f.$$

Thus, $\{\pi T_1, \pi T_2, \ldots, \pi T_f\}$ is also a complete set of orthogonal tableaux.

<u>Definition</u>: If $S_1 = \{T_1, T_2, \ldots, T_f\}$ and $S_2 = \{T_1', T_2', \ldots, T_f'\}$, then S_1 and S_2 are said to be <u>similar</u> if there exists a permutation π such that $\{e_1, e_2, \ldots, e_f\} = \{\pi e_1' \pi^{-1}, \pi e_2' \pi^{-1}, \ldots, \pi e_f' \pi^{-1}\}$.

It is clear that similarity is an equivalence relation.

The problem of finding complete sets of orthogonal tableaux now becomes: Determine for which frames complete sets of orthogonal tableaux exist, and for these frames, classify the sets up to similarity. If $\{T_1, T_2, \ldots, T_f\}$ is a complete set of orthogonal tableaux, let π be the permutation that takes T_1 into the first standard tableau. Then $\{\pi T_1, \pi T_2, \ldots, \pi T_f\}$ is a complete set of orthogonal tableaux which contains the first standard tableau, and is similar to $\{T_1, T_2, \ldots, T_f\}$. When we are constructing a complete set of orthogonal tableaux, we may assume, without loss of generality, that the set will contain the first standard tableau.

Next we consider the group algebra $C(S_n)$. For simplicity, we will denote it by O. For a given frame, we know that the minimal left ideals $Oe_1, Oe_2, \ldots, Oe_{n!}$ span a two-sided ideal which is isomorphic to the ring of all fxf matrices over the complex numbers, where f is the number of standard tableaux for the frame. Since the left ideals Oe_i are minimal, two left ideals Oe_i and Oe_j either intersect trivially, or they are equal. The following theorem gives necessary and sufficient conditions for the two left ideals to be equal.

<u>Theorem 3.1</u>: If T and T' are tableaux belonging to the same frame, then $Oe = Oe'$ if and only if there exists a vertical permutation q for T such that qT and T' are equivalent.

<u>Proof</u>: If there exists a vertical permutation q for T such that qT and T' are equivalent, then $qeq^{-1} = e'$. But $qeq^{-1} = \text{sgn}(q^{-1})qe$ and $Oq = O$. Thus, $Oe = Oe'$. If $Oe = Oe'$, then $e' \varepsilon Oe$, so $e' = xe$ for some $x \varepsilon O$. Then $e'e = xee = xe = e' \neq 0$. By Theorems 2.4-2.6, there exists a horizontal permutation p for T and a vertical permutation q_1 for T such that $T' = pq_1T$. Thus, $\text{sgn}(q_1^{-1})pq_1ep^{-1} = pq_1eq_1^{-1}p^{-1} = e' = e'e = pq_1eq_1^{-1}p^{-1}e = \text{sgn}(q_1^{-1})pq_1e$. Therefore, $ep^{-1} = e$, and so $pep^{-1} = e = p^{-1}ep$. Since p^{-1} is a horizontal permutation for T and $p^{-1}ep = e$, by the remarks following the definition of equivalent, p^{-1} can only interchange columns of T. Thus, q_1 is also a vertical permutation for $p^{-1}T$. By the remarks preceding Theorem 2.3, there exists a vertical permutation q for T such that $q_1 = p^{-1}qp$. Then $pq_1 = qp$, so $T' = qpT$. Therefore, $e' = qpep^{-1}q^{-1} = qeq^{-1}$. Thus, T' is equivalent to qT.

Next we will prove a theorem that is very useful in constructing complete sets of orthogonal tableaux. We begin with a lemma:

<u>Lemma 3.2</u>: Suppose T_1 and T_2 belong to the frame $(m_1, m_2, \ldots, m_r)$, and $e_1 e_2 \neq 0$. Let i be an integer such that $1 \leq i \leq r$. Then in each of the columns 1 through m_i of T_1, there exists exactly one number from row i of T_2.

<u>Proof</u>: By Theorems 2.4 and 2.5, the numbers in row i of T_2 must occur in different columns of T_1. We proceed by induction on i. Since there are m_1 numbers in the first row of T_2, and the frame $(m_1, m_2, \ldots, m_r)$ has m_1 columns, the result holds for i=1. Assume the result is true for i<k, where $2 \leq k \leq r$. If $m_k = m_1$, then, as above, the result holds for i=k. Suppose $m_k < m_1$. Since T_1 and T_2 have the same frames, the induction hypothesis implies that the columns m_k+1 through m_1 in T_1 must be occupied by numbers from rows 1 through j in T_2, where j is the largest integer such that $m_k < m_j$. Thus, the numbers in row k in T_2 must occur in columns 1 through m_k in T_1. Since there are m_k numbers in row k of T_2, and these numbers must occur in different columns of T_1, we have that the result is true for i=k. By the principle of Mathematical Induction, the result is true for $i = 1, 2, \ldots, r$.

<u>Theorem 3.3</u>: Suppose T_1 and T_2 are two tableaux belonging to the frame $(m_1, m_2, \ldots, m_r)$, $m_1 + m_2 + \ldots + m_r = n$. Suppose n occurs in row i, column m_i in both T_1 and T_2, where

$m_i > m_{i+1}$. Let T_1' and T_2' be the two tableaux belonging to $(m_1,\ldots,m_i-1,\ldots,m_r)$ obtained by removing n from T_1 and T_2, respectively. Then $e_1e_2 = 0$ if and only if $e_1'e_2' = 0$.

<u>Proof</u>: If $e_1'e_2' = 0$, then by Theorems 2.4-2.6, there exists two numbers which occur in one column of T_1' and in one row of T_2'. But then this must also hold in T_1 and T_2. Thus, $e_1e_2 = 0$ by Theorem 2.5. If $e_1'e_2' \neq 0$, then the numbers in row j of T_2' must occur in different columns of T_1' (by Theorems 2.4-2.6), for $j = 1,2,\ldots,r$. By Lemma 3.2, the numbers in row i of T_2' occur in columns 1 through m_i-1 of T_1'. Since n occurs in row i of T_2 and column m_i of T_1, the numbers in row i of T_2 occur in different columns in T_1. If $j \neq i$, we know by above that the numbers in row j of T_2 occur in different columns in T_1. By Theorems 2.4 and 2.6, $e_1e_2 \neq 0$.

The next theorem states that a complete set of orthogonal tableaux is determined by its rows.

<u>Theorem 3.4</u>: If $\{T_1,T_2,\ldots,T_f\}$ and $\{T_1',T_2',\ldots,T_f'\}$ are complete sets of orthogonal tableaux for the frame (m) such that $P_i = P_i'$ for $i = 1,2,\ldots,f$, then $e_i = e_i'$ for $i = 1,2,\ldots,f$.

Proof: Let $\kappa = n!/f$. Then $e_i = \kappa^{-1}P_iQ_i$ and $e_i' = \kappa^{-1}P_iQ_i'$ for $i = 1,2,\ldots,f$. Since $Q_iP_j = \delta_{ij}Q_iP_j$ by Theorem 2.5, $e_ie_j' = \delta_{ij}e_ie_j'$. Since $Q_i'P_j = \delta_{ij}Q_i'P_j$ by Theorem 2.5, $e_i'e_j = \delta_{ij}e_i'e_j$. Let e_0 be the identity for the ideal corresponding to (m). By Theorem 2.16,

$$\sum_{i=1}^{f} e_i' = e_0 = \sum_{i=1}^{f} e_i.$$

Thus, $e_j = e_je_0 = \sum_{i=1}^{f} e_je_i' = \sum_{i=1}^{f} \delta_{ij}e_je_i' = e_je_j'$, and $e_j' = e_0e_j' = \sum_{i=1}^{f} e_ie_j' = \sum_{i=1}^{f} \delta_{ij}e_ie_j' = e_je_j'$, for each $j = 1,2,\ldots,n$.

A very important concept in the theory of group representations is that of the character of an irreducible representation. The character of an irreducible representation of S_n is the system of numbers which are the traces of the matrices corresponding to the permutations of S_n [4, p. 81]. Since the character of an irreducible representation of S_n is independent of the basis with which we choose to write the matrices, one might believe that it could not be used to find complete sets of orthogonal tableaux. The character can be used, however, to derive certain restrictions with which a complete set of orthogonal tableaux must comply. For example, if $\{T_1,T_2,\ldots,T_f\}$ is a complete set of orthogonal tableaux for a given frame, and $\pi = (cd)$ is a transposition, then $tr(\pi) = \sum_{i=1}^{f} [\pi]_{ii}$ in this irreducible representation, where $[\pi]_{ii}e_i = e_i\pi e_i$. Now if c and d occur together in a row of T_i, then π is a horizontal permutation for T_i. In this case, $e_i\pi e_i = e_i$ by Theorem 2.6, and so $[\pi]_{ii} = 1$. If c and d occur together in a column of T_i, then π is a vertical permutation for T_i. In this case $e_i\pi e_i = -e_i$ by Theorem 2.6, and so $[\pi]_{ii} = -1$. Suppose c and d do not occur together in a row of T_i and do not occur together in a column of T_i. Then one can not get from πT_i to T_i by first applying a horizontal permutation to πT_i and then applying a vertical permutation to the resulting tableau. Thus, $e_i\pi e_i = 0$ by Theorems 2.4 and 2.5 and the remarks preceding Theorem 2.4. In this case, $[\pi]_{ii} = 0$. Let R_{cd} be the number of tableaux from $\{T_1, T_2, \ldots, T_f\}$ in which c and d occur together in a row. Let C_{cd} be the number of tableaux from $\{T_1, T_2, \ldots, T_f\}$ in which c and d occur together in a column. Then $R_{cd} - C_{cd} = tr(cd)$. But for a given frame, the trace of a transposition is some constant t_2 that can be calculated using only the shape of the frame [6, p. 69]. Thus, given any complete set of orthogonal tableaux for this frame, $R_{cd} - C_{cd} = t_2$ for all c and d. Similar arguments can be used for other permutations, but the arguments get rather involved if one considers permutations of order greater than three.

Next we will reduce the problem of finding complete sets of orthogonal tableaux to that of solving a very special diophantine equation. Let $T_1, T_2, \ldots, T_{n!}$ be the distinct tableaux for the frame (m). Let f be the number of standard tableaux for (m). For $i = 1,2,\ldots,n!$, let x_i be a variable which is allowed to assume the values one and zero, only. Let e_0 denote the identity for the ideal corresponding to (m). Suppose we can choose f of the x_i equal to 1 and the remaining x_i equal to 0 such that

$$\sum_{i=1}^{n!} x_i e_i = e_0.$$

Given k and j, $e_k e_j = \varepsilon_{kj} e_{kj}$ by Theorem 2.7. Also, $e_k e_i e_j = \varepsilon_{ki} e_{ki} e_j = \varepsilon_{ki} \varepsilon_{ij} e_{kj}$ for $i = 1,2,\ldots,n!$ Thus,

$$\varepsilon_{kj} e_{kj} = e_k e_0 e_j = \sum_{i=1}^{n!} x_i e_k e_i e_j = \sum_{i=1}^{n!} x_i \varepsilon_{ki} \varepsilon_{ij} e_{kj}.$$

Therefore, we have

$$\sum_{i=1}^{n!} \varepsilon_{ki} \varepsilon_{ij} x_i = \varepsilon_{kj}, \quad k,j = 1,2,\ldots,n! \qquad (3.1)$$

Let S be a complete set of orthogonal tableaux for (m). By Theorem 2.16, the idempotents derived from the tableaux in S sum to e_0. Thus, the following is a solution to (3.1):

$$x_i = \begin{cases} 0 & \text{if } T_i \notin S \\ 1 & \text{if } T_i \in S \end{cases}.$$

Now (3.1) is $(n!)^2$ equations in $n!$ unknowns. How many of these equations are independent? Suppose T_1, $T_2,\ldots,T_f$ are the standard tableaux for (m). We claim that the e_{ij}, $i,j = 1,2,\ldots,f$, are linearly independent. Suppose $\sum_{i,j=1}^{f} \alpha_{ij}e_{ij} = 0$ for some complex numbers α_{ij}, $i,j = 1,2,\ldots,f$. By Theorem 2.15, $\sum_{i=1}^{f} \alpha_{ij}e_{ij} = 0$ for $j = 1,2,\ldots,f$. By Theorems 2.7 and 2.14,

$$0 = e_f \sum_{i=1}^{f} \alpha_{ij}e_{ij} = \sum_{i=1}^{f} \alpha_{ij}\delta_{if}e_{fj} = \alpha_{fj}e_{fj},\ j = 1,2,\ldots,f.$$

Thus, $\alpha_{fj} = 0$ for each $j = 1,2,\ldots,f$. Similarly, successive left multiplications of $\sum_{i=1}^{f-1} \alpha_{ij}e_{ij}$ by $e_{f-1},e_{f-2},\ldots,e_1$ yield $\alpha_{ij} = 0$ for each $i,j = 1,\ldots,f$. Since the ideal corresponding to (m) has dimension f^2 as a vector space over the complex numbers, the e_{ij}, $i,j = 1,2,\ldots,f$, must span this ideal (as a vector space over the complex numbers). Now $e_{\ell k}e_{jm} = \varepsilon_{kj}e_{\ell m}$ and $e_{\ell k}e_i e_{jm} = \varepsilon_{ki}e_{\ell i}e_{jm} = \varepsilon_{ki}\varepsilon_{ij}e_{\ell m}$ by Theorem 2.7. Thus, (3.1) has at most f^2 linearly independent equations.

We can view (3.1) in a slightly different light. Let X be the $n!\times n!$ matrix with $[X]_{ij} = \delta_{ij}x_j$, $i,j = 1,2,\ldots,n!$. Let B be the $n!\times n!$ matrix with $[B]_{ij} = \varepsilon_{ij}$,

$i,j = 1,2,\ldots,n!$. By Theorem 2.2, $\kappa e_0 = \sum_{i=1}^{n!} e_i$, where $\kappa = n!/f$. This implies $\kappa\varepsilon_{kj}e_{kj} = e_k\kappa e_0 e_j = \sum_{i=1}^{n!} e_k e_i e_j = \sum_{i=1}^{n!} \varepsilon_{ki}\varepsilon_{ij}e_{kj}$. Therefore, we have

$$\sum_{i=1}^{n!} \varepsilon_{ki}\varepsilon_{ij} = \kappa\varepsilon_{kj}. \tag{3.2}$$

Thus, $B^2 = \kappa B$. Since $x_i = 0$ or 1 for each i, $X^2 = X$. Thus, (3.1) becomes $BXB = B$, with $B^2 = \kappa B$ and $X^2 = X$.

For a certain class of frames, we can use (3.1) to show that a complete set of orthogonal tableaux partitions the set of all tableaux. First we need some preliminaries.

Definition: For a given tableau T_i, we define $S(T_i) = \{T_k\colon\ \varepsilon_{ik} \neq 0 \text{ and } \varepsilon_{ki} \neq 0\}$. If E is a collection of tableaux, we define $S(E) = \bigcup_{T_i \varepsilon E} S(T_i)$. We will also denote $S(S(T_i))$ by $S^2(T_i)$.

Lemma 3.5: If T_1 and T_2 are two tableaux belonging to the frame $(n-k,k)$, $n-k\geq k>0$, then $T_1 \varepsilon S(T_2)$ if and only if $\varepsilon_{12}\varepsilon_{21} = 1$.

Proof: If $\varepsilon_{12}\varepsilon_{21} = 1$, then $\varepsilon_{12} \neq 0$ and $\varepsilon_{21} \neq 0$, so by definition $T_1 \varepsilon S(T_2)$. If $T_1 \varepsilon S(T_2)$, then $\varepsilon_{12} \neq 0$ and $\varepsilon_{21} \neq 0$. Thus, $e_1e_2 \neq 0$ and $e_2e_1 \neq 0$. Since $e_2e_1 \neq 0$, by Theorems 2.4 and 2.5 and the remarks preceding Theorem 2.4, we can get from T_1 to T_2 by first applying a horizontal permutation to T_1, and then applying a vertical permutation to the resulting tableau. By Theorem 2.6 and the remarks following it, ε_{21} is the sign of this vertical permutation. Let t be the cardinality of $\{r: r$ is in row 2 of T_1 and r is in row 2 of $T_2\}$. By above, $\varepsilon_{21} = (-1)^{k-t}$. Similarly, $\varepsilon_{12} = (-1)^{k-t}$. Thus, $\varepsilon_{12}\varepsilon_{21} = 1$.

Lemma 3.6: If T_1 and T_2 are two tableaux belonging to the frame $(m_1, m_2, ..., m_r)$ with $m_3 = 1$, then $T_1 \varepsilon S(T_2)$ if and only if $\varepsilon_{12}\varepsilon_{21} = 1$.

Proof: As above, $\varepsilon_{12}\varepsilon_{21} = 1$ implies $T_1 \varepsilon S(T_2)$. Suppose $T_1 \varepsilon S(T_2)$. Then $e_1e_2 \neq 0$ and $e_2e_1 \neq 0$. By Lemma 3.2, the numbers in rows 3 through r in T_1 must occur in column 1 of T_2, and the numbers in rows 3 through r in T_2 must occur in column 1 of T_1. Let C_1 denote the set of numbers in the first column of T_1 and C_2 denote the set of numbers in the first column of T_2. Then C_1 and C_2 differ by at most two numbers. Suppose $C_1 = C_2$. If q is the vertical permutation which takes the first column of T_1 into that of T_2, then q^{-1} is the vertical permutation which takes the first column of T_2 into that of T_1. Let t be the cardinality of $\{r: r$ is in row 2 of T_1 and T_2, and $r \notin C_1\}$. Then $\varepsilon_{12} = (-1)^{m_2-1-t}\operatorname{sgn}(q^{-1})$ and $\varepsilon_{21} = (-1)^{m_2-1-t}\operatorname{sgn}(q)$. Thus, $\varepsilon_{12}\varepsilon_{21} = 1$. Suppose C_1 and C_2 differ by exactly two numbers, say $c_1, d_1 \varepsilon C_1 \sim C_2$ and $c_2, d_2 \varepsilon C_2 \sim C_1$. Since there is a horizontal permutation for T_1 which moves the numbers of T_1 into the correct columns they occupy in T_2, c_1 and d_1 occur in the first two rows of T_1. Similarly, c_2 and d_2 occur in the first two rows of T_2. Thus, rows 3 through r of the first column of T_1 contain the same numbers as rows 3 through r of the first column of T_2. Let q be the permutation which takes the numbers in rows 3 through r of T_1 into their correct positions in T_2. Let t be the cardinality of $\{r:\ r$ is in row 2 of T_1 and $T_2\}$. Then $\varepsilon_{12}=(-1)^{m_2-t}\operatorname{sgn}(q^{-1})$ and $\varepsilon_{21} = (-1)^{m_2-t}\operatorname{sgn}(q)$. Thus, $\varepsilon_{12}\varepsilon_{21} = 1$. Suppose C_1 and C_2 differ by exactly one number, say $c_1 \varepsilon C_1 \sim C_2$ and $c_2 \varepsilon C_2 \sim C_1$. Since there is a horizontal permutation for T_1 which moves the numbers of T_1 into the correct columns they occupy in T_2, c_1 and c_2 must occur together in the same row of T_1, say row i. Similarly, c_1 and c_2 occur together in row j of T_2. Suppose $i = j$. Then there exists a vertical permutation q which takes the remaining r-1 numbers in column 1 of T_1 into their correct positions in T_2. Note that q^{-1} takes these r-1 numbers in column 1 of T_2 into their correct positions in T_1. Let t be the cardinality of $\{r:\ r$ is in row 2 of T_1 and T_2, and $r \notin C_1\}$. Then $\varepsilon_{21} = (-1)^{m_2-i-t}\operatorname{sgn}(q)$ and $\varepsilon_{12} = (-1)^{m_2-i-t}\operatorname{sgn}(q^{-1})$. Thus, $\varepsilon_{12}\varepsilon_{21} = 1$. Suppose $i \neq j$. Let x be the number in row j, column 1 of T_1. There exists a vertical permutation for T_1 which takes the numbers in T_1 into the correct rows they occupy in T_2 (by Theorems 2.4 and 2.5 and the remarks preceding Theorem 2.4). If q is the vertical permutation for T_1 which takes the numbers in column 1 of $(c_1x)T_1$ into the correct rows they occupy in T_2, then $(c_2x)q^{-1}$ is the vertical permutation for T_2 which takes the numbers in column 1 of T_2 into the correct rows they occupy in T_1. Let t be the cardinality of $\{r: r \text{ is in row 2 of } T_1 \text{ and } T_2, \text{ and } r \notin C_1\}$. Then

$\varepsilon_{21} = (-1)^{m_2-1-t} \operatorname{sgn}((c_2x)q^{-1})$ and

$\varepsilon_{12} = (-1)^{m_2-1-t} \operatorname{sgn}(q(c_1x))$. Thus, $\varepsilon_{12}\varepsilon_{21} = 1$. The result is proved.

We note that Lemmas 3.5 and 3.6 also hold for the conjugates of these frames. Given any other frame, however, there exists tableaux T_1 and T_2 such that $T_1 \varepsilon S(T_2)$ and $\varepsilon_{12}\varepsilon_{21} = -1$. To see this, we note that such a frame will contain the frame (3,3,2) in the upper left hand corner. For the frame (3,3,2), if we let

$$T_1 = \begin{matrix}123\\456\\78\end{matrix} \quad \text{and} \quad T_2 = \begin{matrix}375\\861\\42\end{matrix} ,$$

then $\varepsilon_{12} = 1$ while $\varepsilon_{21} = -1$. Thus, we have $\varepsilon_{ij}\varepsilon_{ji} = 1$ or 0 for all tableaux T_i, T_j of a frame if and only if the frame does not contain the frame (3,3,2) in the upper left hand corner.

If we consider a frame not containing the frame (3,3,2), and let $k = j$ in (3.1), then we have

$$\sum_{i=1}^{n!} \varepsilon_{ki}\varepsilon_{ik}x_i = 1, \ k = 1,2,\ldots,n!. \qquad (3.3)$$

By Lemmas 3.5 and 3.6, $\varepsilon_{ki}\varepsilon_{ik}$ is either 1 or 0 for all i. Since each x_i is either 1 or 0, we must have exactly one term $\varepsilon_{ki}\varepsilon_{ik}x_i$ nonzero.

<u>Theorem 3.7</u>: If T_1 and T_2 are tableaux belonging to a frame not containing the frame (3,3,2) and $T_1 \varepsilon S^2(T_2)$, then T_1 and T_2 cannot be together in a complete set of orthogonal tableaux.

<u>Proof</u>: By definition, $T_1 \varepsilon S^2(T_2)$ implies there exists a tableau T_k in $S(T_2)$ such that $T_1 \varepsilon S(T_k)$. Thus, $\varepsilon_{k2}\varepsilon_{2k} = 1$ and $\varepsilon_{1k}\varepsilon_{k1} = 1$. Suppose C is a complete set of orthogonal tableaux for this frame. Let $x_i = 1$ if $T_i \varepsilon C$ and $x_i = 0$ if $T_i \notin C$. By the remarks following (3.1), these x_i satisfy (3.3). Since $\varepsilon_{k2}\varepsilon_{2k} = 1 = \varepsilon_{1k}\varepsilon_{k1}$, the remarks preceding this theorem imply at most one of x_1 and x_2 can be nonzero.

Suppose $\{T_1,T_2,\ldots,T_f\}$ is a complete set of orthogonal tableaux for a frame not containing a (3,3,2). If we let k=j in (3.2), then we have $\sum_{i=1}^{n!} \varepsilon_{ki}\varepsilon_{ik} = \kappa$, where $\kappa = n!/f$. By the remarks following Lemma 3.6, $\varepsilon_{ki}\varepsilon_{ik} =$ 0 or 1 for all i and k. Thus, the cardinality of $S(T_k)$ is

κ for all k. By Theorem 3.7, $S(T_k) \cap S(T_j) = \phi$ for $k,j = 1,2,\ldots,f$. Thus, the $S(T_k)$, $k = 1,2,\ldots,f$, partition the set of all tableaux for this frame.

Definition: Let $K(T_i) = \{T_k\colon \varepsilon_{ik} = 0 = \varepsilon_{ki}\}$. By $\sim S^2(T_i)$ we will mean all those tableaux (for a given frame) not contained in $S^2(T_i)$.

Theorem 3.8: $\sim S^2(T_i) \subset K(T_i)$.

Proof: If $\varepsilon_{mi} \neq 0$, then there exists a horizontal permutation p for T_i and a vertical permutation q for T_i such that $pqT_i = T_m$. Let $T_\ell = pT_i$. By Theorem 2.6, $T_\ell \varepsilon S(T_i)$. Let $q' = pqp^{-1}$. Then q' is a vertical permutation for T_ℓ. Also, $q'T_\ell = (pqp^{-1})pT_i = pqT_i = T_m$. Thus, $T_m \varepsilon S(T_\ell)$. Therefore, $T_m \varepsilon S^2(T_i)$. If $\varepsilon_{im} \neq 0$, then there exists a horizontal permutation p for T_m and a vertical permutation q for T_m such that $pqT_m = T_i$. Let $T_\ell = pT_m$. Then $T_m \varepsilon S(T_\ell)$. Let $q' = pqp^{-1}$. Then q' is a vertical permutation for T_ℓ. Also, $q'T_\ell = T_i$. Thus, $T_\ell \varepsilon S(T_i)$. Therefore, $T_m \varepsilon S^2(T_i)$. Thus, if $T_m \not\varepsilon S^2(T_i)$, then $\varepsilon_{mi} = 0 = \varepsilon_{im}$, so $T_m \varepsilon K(T_i)$.

If we are trying to construct a complete set of orthogonal tableaux containing the tableau T, by definition we need only consider those tableaux T' for which ee' = 0 = e'e, i.e., those tableaux in K(T). If the frame for T does not contain a (3,3,2), then by Theorem 3.7, we need only consider those tableaux in $\sim S^2(T)$. By Theorem 3.8, $\sim S^2(T) \subset K(T)$. If $\sim S^2(T) \neq K(T)$, we have reduced the number of tableaux we must consider. This will be the case for any frame that contains either (3,2) or (2,2,1) in the upper left hand corner. To see this for the case in which the frame contains (3,2), let $T_1 = \begin{smallmatrix}123\\45\end{smallmatrix}$, $T_2 = \begin{smallmatrix}235\\41\end{smallmatrix}$, and $T_3 = \begin{smallmatrix}453\\21\end{smallmatrix}$. Then $\varepsilon_{12} = 0 = \varepsilon_{21}$, $\varepsilon_{23} = -1 = \varepsilon_{32}$, and $\varepsilon_{13} = 1 = \varepsilon_{31}$. Thus, $T_3 \varepsilon S(T_2)$ and $T_1 \varepsilon S(T_3)$. Therefore, $T_1 \varepsilon K(T_2)$ and $T_1 \varepsilon S^2(T_2)$. For the case in which the frame contains (2,2,1), we use the conjugates of the above tableaux.

We prove one final result for this section. For a given frame, let f be the number of standard tableaux. Let $\kappa = n!/f$. Let t_1 be the number of horizontal permutations for this frame and t_2 be the number of vertical permutations for this frame. Then we have the following:

<u>Theorem 3.9</u>: For any frame not containing the frame (3,3,2), the cardinality of K(T) is $n! - 2t_1t_2 + \kappa$ for all tableaux T belonging to this frame.

<u>Proof</u>: Let $T' = \pi T$ for some permutation π. By Theorems 2.4-2.6, $e'e \neq 0$ if and only if $\pi = pq$ for some horizontal permutation p for T and some vertical permutation q for T. By Theorems 2.4-2.6, $ee' \neq 0$ if and only if $\pi^{-1} = p'q'$ for

some horizontal permutation p' for T' and some vertical permutation q' for T'. Thus, there are t_1t_2 tableaux T' such that $e'e \neq 0$, and there are t_1t_2 tableaux T' such that $ee' \neq 0$. By the remarks following Theorem 3.7, there are κ tableau T' such that $ee' \neq 0$ and $e'e \neq 0$. Therefore, the number of tableaux in K(T) is $n! - 2t_1t_2 + \kappa$.

We end this section by relating the problem of finding complete sets of orthogonal tableaux to graph theory. For the n! tableaux for a given frame (m), we form a graph G as follows:

Let i denote a vertex representing T_i. The vertices i and j are adjacent if and only if $e_ie_j = 0 = e_je_i$.

The problem of finding all complete sets of orthogonal tableaux for (m) is equivalent to finding all f-cliques of G, where f is the number of standard tableaux for (m).

B. Flat Tableaux

By a flat tableau, we mean any tableau belonging to a frame of the form $(n-k, k)$, $n-k\geq k>0$. We have the following special terminology for flat tableaux:

Definition: Let T be a tableau corresponding to the frame $(n-k, k)$, $n-k\geq k>0$. We say the number r is in the bottom of T if r is in the second row of T. We say the number r is in the middle of T if r is in the first row of T and in the first k columns of T. If $n-k>k$, we say the number r is in the end of T if r is contained in columns $k+1$ through $n-k$ of T.

It is clear that if r is contained in the bottom of a tableau T, we can find a tableau T' which is equivalent to T in which r is in row 2 and column k. Also, if r is contained in the end of T, we can find a tableau T' which is equivalent to T in which r is in row 1 and column $n-k$.

For the frame $(n-k,k)$, $n-k>k>0$, Theorem 2.12 becomes $f(n-k,k) = f(n-k-1,k) + f(n-k,k-1)$. For a complete set of orthogonal tableaux for this frame, Theorem 3.3 implies that a given number occurs in the bottom of at most $f(n-k,k-1)$ tableaux and in the end of at most $f(n-k-1,k)$ tableaux.

For the frame $(n-k,k)$, $n = 2k$, Theorem 2.12 becomes $f(n-k,k) = f(n-k,k-1)$. Given any tableau T in this frame, we can find a tableaux T' which is equivalent to T, in which n occurs in row 2 and in column k. Thus, given any complete set of orthogonal tableaux for this frame, we can find a set similar to this one in which n occurs in row 2 and column k of every tableau.

Suppose we have a complete set of orthogonal tableaux for the frame $(n-k,k)$, $n-k>k>0$. The next theorem relates the number of times a given number appears in the end to the number of times it appears in the bottom.

Theorem 3.10: Let $\{T_1,T_2,\ldots,T_f\}$ be a complete set of orthogonal tableaux for the frame $(n-k,k)$, $n-k>k>0$. For a fixed number r, let b equal the number of tableaux from $\{T_1,T_2,\ldots,T_f\}$ in which r occurs in the bottom, and let e equal the number of tableaux from $\{T_1,T_2,\ldots,T_f\}$ in which r occurs in the end. Let t equal the trace of a transposition in this representation. Then $(n-1)t = e + (2k-n)b + (n-k-2)f$.

Proof: Let x be the sum of the transpositions containing r. Then $tr(x) = (n-1)t$. Now we compute the trace of x in the particular basis corresponding to $\{T_1,T_2,\ldots,T_f\}$. By the discussion following Theorem 3.4, we know that if s is a number occurring with r in a row of T_i, then $[(rs)]_{ii} = 1$. If s is a number occurring with r in a column of T_i, then $[(rs)]_{ii} = -1$. If, in T_i, s is not in a row with r and not in a column with r, then $[(rs)]_{ii} = 0$. Thus, if r occurs in the end of T_i, it occurs in the top row of T_i with $n-k-1$ other numbers, and it does not occur with any number in a column, so $[x]_{ii} = n-k-1$. If r occurs in the middle of T_i, it occurs with n-k-1 other numbers in the top row of T_i, and it occurs in a column with 1 number, so $[x]_{ii} = n-k-2$. If r occurs in the bottom of T_i, it occurs in the bottom row with k-1 other numbers, and it occurs in a column with 1 number, so $[x]_{ii} = k-2$. Since the number of tableaux in which r occurs in the middle is f-b-e, we have $(n-1)t = (n-k-1)e + (n-k-2)(f-b-e) + (k-2)b = e + (2k-n)b + (n-k-2)f$.

We will end this chapter with some observations about rectangular flat tableaux, i.e., those flat tableaux for which the two rows have equal length. We know that given any complete set of orthogonal tableaux for a frame (n-k,k), n = 2k, we can find a set similar to it in which n occurs in row 2 and column k of each tableau. If we remove n from each of these tableaux, we are left with a complete set of orthogonal tableaux for (n-k,k-1), by Theorem 3.3. Conversely, if we have a complete set of orthogonal tableaux for (n-k,k-1), n = 2k, then adding n to row 2, column k of each tableau yields a complete set of orthogonal tableaux for (n-k,k).

Definition: Let S_1 and S_2 be two complete sets of orthogonal tableaux for the frame (n-k,k-1), n = 2k. Let S_1' and S_2' be the corresponding complete sets of orthogonal tableaux for (n-k,k) obtained by adding n to row 2, column k of each tableau. We say S_1 and S_2 are _equivalent modulo (n-k,k)_ if S_1' and S_2' are similar.

When we classify complete sets of orthogonal tableaux for (n-k,k-1), n = 2k, we need only do so up to equivalence modulo (n-k,k).

IV. CONSTRUCTION OF ORTHOGONAL TABLEAUX

In this chapter, we begin the construction of complete sets of orthogonal tableaux.

A. Standard Tableaux

The first question that comes to mind when searching for complete sets of orthogonal tableaux is:" When are the standard tableaux for a given frame orthogonal?" As we noted in Chapter I, for $n \leq 4$ the standard tableaux are always orthogonal. It is also easy to show that for $n \leq 4$, any complete set of orthogonal tableaux for a given frame is similar to the set of standard tableaux for that frame. For $n > 4$, we have the following:

Theorem 4.1: Given the frame $(m_1, m_2, \ldots, m_r)$, $m_1 + m_2 + \ldots + m_r = n > 4$, the standard tableaux for this frame are orthogonal if and only if $m_2 = 1$.

Proof: Given the frame $(m_1, m_2, \ldots, m_r)$ with $m_2 = 1$, we note that the standard tableaux all have the number 1 in row 1 and column 1. If T_i and T_j are distinct standard tableaux for this frame, then column 1 of T_i must contain a number r such that r is not in column 1 of T_j, otherwise T_i and T_j would be the same tableau, by definition of standard. Thus, r is in row 1 of T_j, since $m_2 = 1$.

Therefore, the numbers 1 and r occur together in a column of T_i and a row of T_j. Since T_i and T_j were arbitrary, the standard tableaux for this frame are orthogonal.

Given the frame $(m) = (m_1, m_2, \ldots, m_r)$ with $m_1+m_2+\ldots+m_r>4$ and $m_2 \geq 2$, we must have either $m_1 \geq 3$ or $m_3 \geq 1$. If $m_1 \geq 3$, then (m) contains the frame $(3,2)$ in the upper left hand corner. Let T_i be any standard tableau for this frame containing $\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}$ in the upper left hand corner (at least one exists). Let T_j be the tableau πT_i with $\pi = (2354)$. Then T_j has $\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}$ in the upper left hand corner, and T_j is a standard tableau for (m). Now $\pi^{-1} = (35)(24)(34)$, where $(35)(24)$ is a horizontal permutation for T_j and (34) is a vertical permutation for T_j. By Theorem 2.6, $e_ie_j \neq 0$. Thus, the standard tableaux in this case are not orthogonal. If $m_3 \geq 1$, then (m) contains the frame $(2,2,1)$ in the upper left hand corner. Applying a similar argument using $\begin{smallmatrix}1&2\\3&4\\5\end{smallmatrix}$ and $\begin{smallmatrix}1&4\\2&5\\3\end{smallmatrix}$, we also find that the standard tableaux are not orthogonal in this case. Thus, if $n>4$ and $m_2 \geq 2$, the standard tableaux are not orthogonal.

To classify all complete sets of orthogonal tableaux for the frame $(m_1, m_2, \ldots, m_r)$ with $m_2 = 1$, we first note that a tableau T for this frame is equivalent to a standard tableau if and only if T has the number 1 in row 1 and column 1. Then we have the following:

Theorem 4.2: If S is a complete set of orthogonal tableaux for the frame $(m) = (m_1, m_2, \ldots, m_r)$ with $m_2 = 1$, then S is similar to the set of standard tableaux for this frame.

Proof: By the remarks following the definition of similarity for complete sets of orthogonal tableaux, S is similar to some set S', where S' contains the first standard tableau T_1. Let s be a number in the first row of T_1, $s \neq 1$. Let t be a number in the first column of T_1, $t \neq 1$. Let $T_k = (1s)T_1$ and $T_j = (1t)T_1$. If T_i is any tableau, then $\varepsilon_{ki} \neq 0$ if and only if the numbers in row 2 through row r of T_i are in column 1 of T_k (Lemma 3.2). In particular, if $\varepsilon_{ki} \neq 0$, then the number 1 occurs in row 1 of T_i. If $\varepsilon_{ij} \neq 0$, then s and t cannot occur together in column 1 of T_i, and the number 1 must occur in column 1 of T_i (Lemma 3.2). Thus, $\varepsilon_{ki}\varepsilon_{ij} \neq 0$ if and only if T_i is equivalent to either T_1 or $(st)T_1$. Now define $x_i = 1$ if $T_i \varepsilon S'$ and $x_i = 0$ if $T_i \not\varepsilon S'$. Then these x_i satisfy (3.1). Also, $\varepsilon_{kj} = 0$. Since $T_1 \varepsilon S'$, we know from above that S' must contain a tableau T_i equivalent to $(st)T_1$, in order to satisfy (3.1) with the above k and j. This holds for all $s \neq 1$ in the first row of T_1 and all $t \neq 1$ in the first column of T_1. We can apply the same argument as above to all these $(st)T_1$. Continuing inductively, we get that for every standard tableau T_i for (m), there exists a tableau T_j in S' such that T_i and T_j are equivalent. Thus, S' is similar to the set of all standard tableaux for (m). Since the relation of similarity for complete sets of orthogonal tableaux is transitive, S is similar to the set of standard tableaux for (m).

B. Computability

In this section, we consider the possibility of solving the problem of finding complete sets of orthogonal tableaux on the computer. As a first method, one might try checking all possible sets of f tableaux to see which ones are orthogonal. This requires brute forcing through $\binom{n!}{f}$ possibilities. For the frame (4,3), this number is on the order of 10^{40}. Such a method is clearly impractical on present day computers. We note, however, that since we need only classify complete sets of orthogonal tableaux up to similarity, we may require that the first standard tableau T_1 be in all sets we search for. Then we need only consider those tableaux which are orthogonal to the first standard tableau. This reduces the number of possibilities we must check considerably. For the frame (4,3), the number of sets of f tableaux we must check reduces to 10^{30}, since $|K(T_1)| = 840$ by Theorem 3.9. Unfortunately, this number of checks is also impossible to do on present day computers. Actually, the number of checks is considerably less than this. When constructing complete sets of orthogonal tableaux, we choose the tableaux one at a time. When a second tableau T_2 is chosen, we need not consider all the remaining tableaux of $K(T_1)$. We need only consider those tableaux in $K(T_1) \cap K(T_2)$. However, such a scheme still remains impractical.

For frames not containing the frame (3,3,2), we know by Theorem 3.7 that if a complete set of orthogonal tableaux contains the first standard tableau T_1, then the remaining tableaux must come from $\sim S^2(T_1)$. By Theorem 3.8, $\sim S^2(T_1) \subset K(T_1)$. For this case of (4,3), $|\sim S^2(T_1)| = 331$, which is considerably less than the 840 tableaux in $K(T_1)$. In this case, the number of sets of f tableaux we must check reduces to 10^{23}. As above, the actual number of checks we must do is considerably less than this. In this case, we are almost in the realm of computability. However, we need only go to (5,3) to once again be beyond the capability of present day computers.

Finally, we note that the run time for all known algorithms for finding complete sets of orthogonal tableaux for a given frame appears to grow exponentially with n. For $n>7$, the programs become impossible to run on present day computers.

C. Classification of (n-2,2)

The use of graph theory provides a particularly nice classification of the complete sets of orthogonal tableaux for the frames (n-2,2), $n>4$. Unfortunately, this classification does not appear to generalize to other frames.

Suppose $\{T_1,T_2,...,T_f\}$ is a complete set of orthogonal tableaux for the frame (n-2,2), $n>4$. We will consider the bottoms of these tableaux as unordered pairs of numbers. Since $e_i e_j = 0$ if $i \neq j$, these unordered pairs must be distinct. Thus, f of the $n(n-1)/2$ unordered pairs of numbers from $\{1,2,...,n\}$ must occur as bottoms for $T_1,T_2,...,T_f$. By Theorem 2.11, $f = n(n-3)/2$. Therefore, there are n unordered pairs not occurring in the bottom of any tableau from $\{T_1,T_2,...,T_f\}$. We form a graph G(n) of n vertices and n edges as follows:

Let $1,2,...,n$ denote the vertices of the graph. For each $i,j = 1,2,...,n$, the vertices i and j are adjacent if and only if the unordered pair $\{i,j\}$ does not occur in the bottom of any tableau from $\{T_1,T_2,...,T_f\}$.

We note that if $\begin{smallmatrix} i & j \\ k & \ell \end{smallmatrix}$ occurs in a tableau from $\{T_1, T_2, \ldots, T_f\}$, then the vertices i and j, j and k, and i and ℓ are adjacent. Thus, there exists a path of length three connecting the vertices k and ℓ. Therefore, G(n) is a graph with n vertices and n edges such that any two nonadjacent vertices are connected by a path of length three. To classify all complete sets of orthogonal tableaux for the frame (n-2,2), we need only find all such graphs.

We begin by noting that G(n) is connected, since given any two vertices of G(n), these vertices are either adjacent or are connected by a path of length three. Also, since G(n) has n edges and n vertices and G(n) is connected, G(n) must contain a cycle [1, pp. 21 and 26]. Now we make the following observation:

Suppose S is a subgraph of a connected graph G. Suppose S has r vertices and q edges, and G has n vertices. Then G must have at least n-r+q edges.

To see this, denote the vertices of S by $\{x_1, x_2, \ldots, x_r\}$. Let C be a component of the subgraph $G - \{x_1, x_2, \ldots, x_r\}$. If C has t vertices, then C has at least t-1 edges. Since G is connected, one of the vertices of C must be adjacent to one of the vertices of S. Thus, G has at least n-r more edges than S.

Let C be an r-cycle of G(n) such that r is minimal, i.e., G(n) contains no t-cycle if $t<r$. Suppose $r>5$. Consider any two vertices of C which are connected by a path of length 2. These two vertices must be connected by a path of length 1 or 3. This implies the existence of a t-cycle with $t\leq 5$, contradicting the minimality of r. Thus, $r\leq 5$. Suppose $r=4$. Consider any two vertices of C which are connected by a path of length 2. These two vertices must be connected by a path P of length 1 or 3. Since r is minimal, this path is not of length 1 and cannot contain an edge from C. Thus, $P\cup C$ is a subgraph of G(n) which contains 6 vertices and 7 edges. By the observation above, this implies G(n) has at least n+1 edges, a contradiction. Therefore, $r=3$ or $r=5$. Suppose $r=5$. Suppose j is a vertex not in C which is adjacent to a vextex i in C. Let k be a vertex in C adjacent to i. Then there exists a path P of length 3 connecting the vertices k and j. If P contains an edge of C, then by minimality of r, $P\cup C$ is a subgraph of G(n) containing 7 vertices and 8 edges. If P does not contain an edge of C, then $P\cup C$ is a subgraph of C(n) containing 8 vertices and 9 edges. In either case, the observation above implies G(n) has at least n+1 edges, a contradiction. Therefore, $r=5$ implies $n=5$. Suppose $r=3$. Let i be a vextex of C. Suppose j and k are vertices not in C which are adjacent to i. The vertices j and k cannot be adjacent, since this would imply the existence of a subgraph of G(n) which contains 5 vertices and 6 edges. Thus, there exists a path P of length 3 connecting the vertices j and k. In all cases, PUC contains at least one more edge than vertex. The observation above implies G(n) has at least n+1 edges, a contradiction. Therefore, each vertex of C can be adjacent to at most one vertex not in C. Let i be a vertex of C, and let j be a vertex not in C which is adjacent to i. Suppose k is a vertex adjacent to j, $k \neq i$. If k is a vertex in C, then the subgraph of G(n) consisting of the vertex j, the edges $\{i,j\}$ and $\{k,j\}$, and C, has 4 vertices and 5 edges, a contradiction. Thus, k cannot be in C. By above, k cannot be adjacent to i, since j is adjacent to i. Thus, there exists a path P of length 3 connecting the vertices k and i. In all cases, PUC contains at least one more edge than vertex. The observation above implies G(n) has at least n+1 edges, a contradiction. Thus, every vertex j not in C is adjacent to exactly one vertex i, and i must be in C. Therefore, if r=3, we have the following possibilities for G(n), n>4:

Thus, up to similarity, there are two complete sets of orthogonal tableaux for (3,2), and one complete set of orthogonal tableaux for (4,2). There are no complete sets of orthogonal tableaux for (n-2,2) if n>6. A complete set of orthogonal tableaux for the frame (n-2,2) corresponding to the graph G(n) is constructed as follows:

If i and j are nonadjacent vertices in G(n), then there exists a path of length 3 connecting i and j, say P: i,k,ℓ,j. We form the tableau with i and j in the bottom by requiring that it contain $\begin{smallmatrix} \ell k \\ ij \end{smallmatrix}$. Since the bottoms of all the tableaux are determined, the tableaux are uniquely determined up to equivalence by Theorem 3.4. The complete sets of orthogonal tableaux for (3,2) and (4,2) are listed in the Appendix.

D. Some Results for (n-3,3)

In this section, we prove that no complete set of orthogonal tableaux exists for (5,3). Also, we begin the classification of complete sets of orthogonal tableaux for (4,3). Since the graph theory technique does not readily extend to the frames (n-3,3), we will give an alternate proof of the nonexistence of a complete set of orthogonal tableaux for (5,2) by using (3.1) and Theorem 3.10. This technique can be used to prove that no complete set of orthogonal tableaux exists for (5,3).

Assume S is a complete set of orthogonal tableaux for the frame (5,2). Without loss of generality, we will assume that S contains the first standard tableau T_1. Define $x_i = 1$ if $T_i \varepsilon S$ and $x_i = 0$ if $T_i \not\varepsilon S$ for each tableau T_i of (5,2). Then these x_i satisfy (3.1).

Let T_j be any tableau with the pair {1,2} in the bottom. Let T_k be any tableau containing $\frac{rs}{67}$, $r,s \varepsilon \{3,4,5\}$. By Lemma 3.2, if T_i is a tableau with $\varepsilon_{ki}\varepsilon_{ij} \neq 0$, then T_i has 1 and 2 in the middle, and one of the pairs {6,7}, {s,6}, {r,7}, or {r,s} in the bottom. If T_i has {1,2} in the middle and either {6,7} or {r,s} in the bottom, then $\varepsilon_{ki}\varepsilon_{ij} = +1$. We will call such a tableau a "tableau of positive charge relative to T_k and T_j". If T_i has {1,2} in the middle and either {6,s} or {r,7} in the bottom, then $\varepsilon_{ki}\varepsilon_{ij} = -1$. We will call such a tableau a "tableau of negative charge relative to T_k and T_j". Also, we note that $\varepsilon_{kj} = 0$. Let T_{k_m}, $m = 1,2,\ldots,6$, denote the six distinct tableaux (up to equivalence) containing $\frac{rs}{67}$, $r,s\varepsilon\{3,4,5\}$. Using these T_{k_m} and T_j given above, we sum (3.1) from m=1 to m=6. If we interchange the order of summations, we obtain:

$$\sum_{i=1}^{n!} \left(\sum_{m=1}^{6} \varepsilon_{k_m i} \varepsilon_{ij} \right) x_i = 0. \tag{4.1}$$

Since $T_1 \varepsilon S$, $x_1=1$. Also, $\varepsilon_{k_m 1}\varepsilon_{1j} = 1$ for $m = 1,2,\ldots,6$. This adds +6 to the left hand side of (4.1). Suppose T_i in S is a tableau of negative charge relative to some T_{k_m} and T_j. Then $\varepsilon_{k_m i}\varepsilon_{ij} = -1$ for exactly two values of m, and $\varepsilon_{k_m i}\ \varepsilon_{ij} = 0$ for the remaining values of m. Thus, each such tableau in S adds -2 to the left hand side of (4.1). Suppose T_i in S contains {1,2} in the middle and {r,s} in the bottom, $r,s \varepsilon \{3,4,5\}$. Then $\varepsilon_{k_m i}\varepsilon_{ij} = 1$ for two values of m, and $\varepsilon_{k_m i}\varepsilon_{ij} = 0$ for the remaining values of m. Thus, each such tableau in S adds +2 to the left hand side of (4.1). Given any tableau T_i with {1,2} in the middle, there exists a T_{k_m} such that T_i is either a tableau of positive charge or a tableau of negative charge relative to T_{k_m} and T_j. Now T_1 in S adds +6 to the left hand side of (4.1). This implies that we need 3 tableaux in S of negative charge relative to the T_{k_m} and T_j, in order to balance (4.1). Each additional tableau in S of negative (positive) charge relative to the T_{k_m} and T_j requires an additional tableau in S of positive (negative) charge relative to the T_{k_m} and T_j, in order to balance (3.1). Therefore, {1,2} occurs in the middle of 2ℓ tableaux in S, $2 \leq \ell \leq 7$. Since there are only ten pairs of numbers which can occur as bottoms in tableaux in which 1 and 2 are in the middle, we have $\ell \leq 5$.

The above argument remains valid for any pair of numbers occurring in the middle of some tableau in S. Thus, if a pair of numbers occurs in the middle of a tableau in S, it occurs in the middle of 4, 6, 8, or 10 tableaux in S. The separate pairs that occur as middles for the tableaux in S induce the following partitions of 14 (there are 14 tableaux in S):

1) 4-4-6
2) 4-10
3) 6-8 ,

where 4-4-6 means that there are two pair of numbers which occur as middles for four tableaux in S, and one pair of numbers which occurs as a middle for 6 tableaux in S, and 4-10 and 6-8 have analogous meanings. We now note that the trace of a transposition in (5,2) is +6 [4, p. 148]. By Theorem 3.10, e = 3b-6, where e is the number of tableaux in S in which a given number appears in the end, and b is the number of tableaux in S in which this same number occurs in the bottom. This implies that a given number occurs in the middle of 0, 4, 8, or 12 tableaux in S. We have the following cases:

1) 4-4-6. If we consider either number from the pair which occurs as a middle for 6 tableaux in S, this number occurs in the middle of 6, 10, or 14 tableaux in S, a contradiction.

2) 4-10. If we consider either number from the pair which occurs as a middle for 10 tableaux in S, this number occurs in the middle of 10 or 14 tableaux in S, a contradiction.

3) 6-8. If we consider either number from the pair which occurs as a middle for 6 tableaux in S, this number occurs in the middle of 6 or 14 tableaux in S, a contradiction. Thus, no such S exists for (5,2).

Unfortunately, the argument for (5,3) is somewhat more involved, due to the increased number of cases that must be considered.

Theorem 4.3: No complete set of orthogonal tableaux exists for (5,3).

Proof: Suppose S is a complete set of orthogonal tableaux for (5,3). By Theorem 2.11, f = 28. Since the bottom of a tableau for (5,3) contains 3 numbers, one of the numbers from {1,2,...,8} must occur in the bottom of at least 11 tableaux in S. Without loss of generality, this number is 1. Let b equal the number of tableaux in S in which 1 occurs in the bottom. Let e equal the number of tableaux in S in which 1 occurs in the end. Let m equal the number of tableaux in S in which 1 occurs in the middle. Since the trace of a transposition in (5,3) is 10 [4, p. 149], $e = 2b-14$ by Theorem 3.10. Since $m = 28-b-e$ and $b \geq 11$, $m=0$, 3, 6, or 9. Suppose $m=0$. Then $b=14$. If we apply the permutation (18) to S, we obtain 14 orthogonal tableaux with 8 in the bottom. Without loss of generality, we may assume the 8 occurs in column 3 of each of these 14 tableaux. By Theorem 3.3, if we remove the 8 from these 14 tableaux, we will be left with a complete set of orthogonal tableaux for (5,2), a contradiction. Therefore, $m = 3$, 6, or 9.

For the 8! tableaux of (5,3), let $x_i = 1$ if $T_i \in S$ and $x_i = 0$ if $T_i \notin S$. Then these x_i satisfy (3.1). Let T_j be any tableau with $\{1,b,c\}$ in the bottom. Let T_{k_m}, $m = 1,2,\ldots,6$, denote the 6 distinct tableaux (up to equivalence) with $\{x,y,z\}$ in bottom and $\{c,r,s\}$ in the middle with $1,b \notin \{r,s,x,y,z\}$. Then $\varepsilon_{k_m j} = 0$ for each m. Using these T_{k_m} and T_j, we sum (3.1) from m=1 to m=6. If we interchange the order of summations, we obtain:

$$\sum_{i=1}^{n!} \left(\sum_{m=1}^{6} \varepsilon_{k_m i} \varepsilon_{ij} \right) x_i = 0. \tag{4.2}$$

Table 1 (given at the end of this proof) gives a summary of those types of tableaux T_i (up to equivalence) for which $\sum_{m=1}^{6} \varepsilon_{k_m i} \varepsilon_{ij} \neq 0$. We will use T_{k_1} and column 1 of Table 1 to explain how Table 1 is interpreted. T_{k_1} contains $\begin{smallmatrix} \text{crs} \\ \text{xyz} \end{smallmatrix}$. Column 1 gives the 8 possible bottoms for a tableau T_i for which $\varepsilon_{k_1 i} \neq 0$. If T_i contains {x,r,s}, {x,z,r}, {x,y,s}, or {x,y,z} in the bottom, then T_i must contain {l,b,c} in the middle if $\varepsilon_{ij} \neq 0$. This is indicated by $\begin{smallmatrix} \text{lbc} \\ \cdots \end{smallmatrix}$ at the far left of the table. If T_i contains one of the remaining bottoms in column 1, then T_i must contain $\begin{smallmatrix} \text{lb.} \\ \text{..c} \end{smallmatrix}$ (up to equivalence) if $\varepsilon_{ij} \neq 0$. This is also indicated at the far left of the table. The column headed "$\varepsilon_{ki} \varepsilon_{ij}$" gives the sign of $\varepsilon_{k_m i} \varepsilon_{ij}$ for each m and for each type of T_i. Thus, if T_i contains {l,b,c} in the middle and {x,r,s} in the bottom, then $\varepsilon_{k_1 i} \varepsilon_{ij} = -1$ and $\varepsilon_{k_2 i} \varepsilon_{ij} = -1$, but $\varepsilon_{k_m i} \varepsilon_{ij} = 0$ if m>2. Thus, $\sum_{m=1}^{6} \varepsilon_{k_m i} \varepsilon_{ij} = -2$ for this T_i. This is indicated in the last column of Table 1, i.e., for a given tableau T_i for which $\varepsilon_{k_m i} \varepsilon_{ij} \neq 0$ for some m, the column headed by "TOTAL" gives $\sum_{m=1}^{6} \varepsilon_{k_m i} \varepsilon_{ij}$. For all other T_i not listed in this table, this sum is zero.

By (3.1), if we sum $\varepsilon_{k_m i}\varepsilon_{ij}$ over all T_i in S, we must get zero (for each m). This implies that if $\varepsilon_{k_m i}\varepsilon_{ij} = 1$ for some tableau T_i in S, then there exists a tableau T_ℓ in S such that $\varepsilon_{k_m \ell}\varepsilon_{\ell j} = -1$. We denote this by saying "the charges of the tableaux T_i in S must balance in each column of Table 1". Also, if we sum $\sum_{m=1}^{6} \varepsilon_{k_m i}\varepsilon_{ij}$ over all T_i in S, we must get zero.

Suppose T is the tableau $\begin{matrix}\text{lbcrs}\\ \text{xyz}\end{matrix}$, and $T \varepsilon S$. If we let T_{k_m} and T_j be as in (4.2), then T adds -6 to the left hand side of (4.2). Thus, there are at least 3 tableaux T_i in S for which $\sum_{m=1}^{6} \varepsilon_{k_m i}\varepsilon_{ij} = +2$. This implies {1,b} occurs in the middle of at least 4 tableaux in S. This argument remains valid for any pair of numbers. Thus, we have that if a pair of numbers occurs in the middle of a tableau in S, it occurs in the middle of at least 4 tableaux in S. Therefore, we cannot have m=3. Thus, m = 6 or 9.

Next we form Table 2. Let T_j be any tableau containing {1,b,c} in the bottom. Let $T_{k'_m}$, m = 1,2,...,6, denote the 6 tableaux containing {x,y,z} in the bottom and $\{b,c,t_1\}$ in the middle with $1 \notin \{t_1,x,y,z\}$. Let $T_{k'_m}$, m = 7,8,...,12, denote the 6 tableaux containing {x,y,z} in the bottom and $\{b,c,t_2\}$ in the middle with $1 \notin \{t_2,x,y,z\}$ and $t_1 \neq t_2$. For a fixed value of t ($t=t_1$ or $t=t_2$), Table 2 is interpreted the same way as Table 1, with one exception. The column headed "TOTAL" gives $\sum_{m=1}^{12} \varepsilon_{k'_m i}\varepsilon_{ij}$ for the indicated T_i. Using these $T_{k'_m}$ and T_j, we sum (3.1) from m=1 to m=12 to obtain:

$$\sum_{i=1}^{n!} \left(\sum_{m=1}^{12} \varepsilon_{k'_m i} \varepsilon_{ij} \right) x_i = 0. \tag{4.3}$$

Table 2 is a summary of the tableaux T_i for which $\sum_{m=1}^{12} \varepsilon_{k'_m i} \varepsilon_{ij} \neq 0.$

Let p be the cardinality of {i: i occurs with 1 in the middle of some tableau of S}.

Suppose m=6. We know from above that if a pair of numbers occurs in the middle of some tableau in S, then it occurs in the middle of at least 4 tableaux in S. Thus, $p \leq 3$. Let T be the tableau $\begin{smallmatrix} 1bcrs \\ xyz \end{smallmatrix}$, and suppose $T \varepsilon S$. Then no tableau in S can contain $\begin{smallmatrix} 1.. \\ .bc \end{smallmatrix}$. Let $t_1 = r$ and $t_2 = s$. Let $T_{k'_m}$ and T_j be as in (4.3). Then T in S adds −12 to the left hand side of (4.3). Let T_i be an allowable tableau in S (T_i does not contain $\begin{smallmatrix} 1.. \\ .bc \end{smallmatrix}$) for which $\varepsilon_{k'_m i} \varepsilon_{ij} = 1$ for some m. By Table 2, we see that $\sum_{m=1}^{12} \varepsilon_{k'_m i} \varepsilon_{ij} = +2$. Thus, we need at least 6 such tableaux to balance (4.3). This implies $m \geq 7$, a contradiction. Therefore, m=9. We know from above that if a pair of numbers occurs in the middle of some tableau in S, then it occurs in the middle of at least 4 tableaux in S. Thus, $2 \leq p \leq 4$.

Suppose p=2. Assume S contains the first standard tableau. Then {1,2,3} occurs as a middle for exactly

9 tableaux in S, and 1 does not occur in the middle of any other tableau in S. Let x=6, y=7, and z=8. Let b=2 and c=3. Let $t_1=4$ and $t_2=5$. Let $T_{k'_m}$ and T_j be as in (4.3). The first standard tableau adds −12 to the left hand side of (4.3). By Table 2, S must contain 6 tableaux T_i with {1,2,3} in the middle such that $\sum_{m=1}^{12} \varepsilon_{k'_m i}\varepsilon_{ij} = +2$, in order to balance (4.3). Also, by Table 2, we see that these 6 tableaux must have {4,6,7}, {4,6,8}, {4,7,8}, {5,6,7}, {5,6,8}, and {5,7,8} as bottoms. Now let $r=t_1$ and $s=t_2$. Let T_{k_m} and T_j be as in (4.2). The first standard tableau adds −6 to the left hand side of (4.2). The above 6 tableaux add +12 to the left hand side of (4.2). Thus, we need 3 tableaux T_i with {1,2,3} in the middle such that $\sum_{m=1}^{6} \varepsilon_{k_m i}\varepsilon_{ij} = -2$, in order to balance (4.2). In this case, we have m=10, a contradiction. Thus, $p\neq 2$.

Suppose p=3. Without loss of generality, we may assume that {1,2,3}, {1,2,4}, and {1,3,4} occur as middles for x_1, x_2, and x_3 tableaux in S, respectively, with $x_1 + x_2 + x_3 = 9$. Suppose $x_1 = 0$. Since p=3, $x_2 \neq 0$ and $x_3 \neq 0$. Suppose T is the tableau $\begin{matrix}124rs\\xyz\end{matrix}$, and $T\varepsilon S$. Let b=2 and c=4. Let T_{k_m} and T_j be as in (4.2). Now there are no tableaux in S containing $\begin{matrix}12.\\..4\end{matrix}$. Also, the bottoms of all tableaux containing {1,2,4} in the middle appear in Table 1. Since $\sum_{m=1}^{6} \varepsilon_{k_m i}\varepsilon_{ij} = -2, +2,$ or -6 for those tableaux T_i

containing {1,2,4} in the middle, we must have that x_2 is even, in order to balance (4.2). Similarly, x_3 is even. Since $x_1 = 0$, we have $x_2 + x_3 = 9$, a contradiction. Thus, $x_1 \neq 0$. Similarly, $x_2 \neq 0$ and $x_3 \neq 0$.

Suppose T is the tableau $\begin{smallmatrix}123rs\\ xyz\end{smallmatrix}$, and $T\varepsilon S$. Let $t_1 = r$ and $t_2 = s$. Let b=2 and c=3. Let $T_{k_m'}$ and T_j be as in (4.3). Then T in S adds -12 to the left hand side of (4.3). Since S does not contain a tableau of the form $\begin{smallmatrix}1..\\ .23\end{smallmatrix}$, we see by Table 2 that there must be at least 7 tableaux in S of the forms $\begin{smallmatrix}123\\ ...\end{smallmatrix}$, $\begin{smallmatrix}1.3\\ .2.\end{smallmatrix}$, or $\begin{smallmatrix}12.\\ ..3\end{smallmatrix}$ (up to equivalence), in order to balance (4.3). Suppose these occur in x_{11}, x_{12}, and x_{13} tableaux in S, respectively. Note that $x_{11} = x_1$ and $x_{11} + x_{12} + x_{13} \geq 7$. Similarly, we have that $\begin{smallmatrix}124\\ ...\end{smallmatrix}$, $\begin{smallmatrix}1.4\\ .2.\end{smallmatrix}$, and $\begin{smallmatrix}12.\\ ..4\end{smallmatrix}$ occur in x_{21}, x_{22}, and x_{23} tableaux in S, respectively, with $x_{22} = x_2$ and $x_{21} + x_{22} + x_{23} \geq 7$. Also, $\begin{smallmatrix}134\\ ...\end{smallmatrix}$, $\begin{smallmatrix}1.4\\ .3.\end{smallmatrix}$, and $\begin{smallmatrix}13.\\ ..4\end{smallmatrix}$ occur in x_{31}, x_{32}, and x_{33} tableaux in S, respectively, with $x_{33} = x_3$ and $x_{31} + x_{32} + x_{33} \geq 7$. Since $x_1 + x_2 + x_3 = 9$, we have $x_{12} + x_{13} + x_{21} + x_{23} + x_{31} + x_{32} \geq 12$. But the tableaux corresponding to x_{ij}, $i \neq j$, are distinct from those corresponding to $x_{k\ell}$, $k \neq \ell$, if $i \neq k$ or $j \neq \ell$, for all such $i,j,k,\ell = 1,2,3$. This implies $m \geq 12$, a contradiction. Thus, $p \neq 3$.

Therefore p=4. Without loss of generality, we may assume that {1,2,3}, {1,2,4}, {1,3,4}, {1,2,5}, {1,3,5}, and $\{1,4,5\}$ occur as middles for x_1, x_2, x_3, x_4, x_5, and x_6 tableaux in S, respectively. We may assume $x_i \geq x_6$ for all i. Thus, $0 \leq x_6 \leq 1$.

The rest of the proof requires considering several cases. To simplify the consideration of these cases, we make the following observations. Suppose S contains exactly one tableau T of the form $\begin{smallmatrix}1bc\\ ...\end{smallmatrix}$ (up to equivalence), say T is $\begin{smallmatrix}1bcrs\\ xyz\end{smallmatrix}$. Let T_{k_m} and T_j be as in (4.2). Then T in S adds -6 to the left hand side of (4.2). Since T is the only tableau in S of the form $\begin{smallmatrix}1bc\\ ...\end{smallmatrix}$, by Table 1 we see that S must contain at least three tableaux of the form $\begin{smallmatrix}1b.\\ ..c\end{smallmatrix}$, in order to balance (4.2). If we interchange the roles of b and c, i.e., if we say S contains the tableau T of the form $\begin{smallmatrix}1cb\\ ...\end{smallmatrix}$ (such a tableau is equivalent to $\begin{smallmatrix}1bc\\ ...\end{smallmatrix}$ provided the numbers occurring in columns 2 and 3 of row 2 are also interchanged), the above argument tells us that S must contain at least three tableaux of the form $\begin{smallmatrix}1c.\\ ..b\end{smallmatrix}$. In the remainder of this proof, we shall refer to the above argument as follows: "Since S contains exactly one tableau of the form $\begin{smallmatrix}1bc\\ ...\end{smallmatrix}$, we see by Table 1 that S must contain at least 3 tableaux of the form $\begin{smallmatrix}1b.\\ ..c\end{smallmatrix}$ and 3 tableaux of the form $\begin{smallmatrix}1.c\\ .b.\end{smallmatrix}$". The above argument can be used to show that if S contains exactly two tableaux of the form $\begin{smallmatrix}1bc\\ ...\end{smallmatrix}$, then by Table 1, S contains at least 2 tableaux of the form $\substack{1.c\\ .b.}$ and 2 tableaux of the form $\substack{1b.\\ ..c}$.

Suppose $x_6 = 1$. Then at least one of x_2, x_3, x_4, or x_5 is 1. Without loss of generality, $x_5 = 1$. Since $x_6 = 1$, we see by Table 1 that S must contain at least 3 tableaux of the form $\substack{14.\\ ..5}$ and 3 tableaux of the form $\substack{1.5\\ .4.}$. Since $x_5 = 1$, we see by Table 1 that S must contain at least 3 tableaux of the form $\substack{13.\\ ..5}$ and 3 tableaux of the form $\substack{1.5\\ .3.}$. The tableaux corresponding to these four forms are distinct, so this implies $m \geq 12$, a contradiction. Thus, $x_6 = 0$.

We know that if a pair of numbers occurs in the middle of some tableau in S, then it occurs in the middle of at least 4 tableaux in S. Thus, $x_2 + x_3 \geq 4$ and $x_4 + x_5 \geq 4$. This implies $0 \leq x_1 \leq 1$.

Suppose $x_1 = 1$. Then $x_2 + x_3 = 4 = x_4 + x_5$. Without loss of generality, $x_2 \leq x_3$. Since $x_1 = 1$, we see by Table 1 that S must contain at least 3 tableaux of the form $\substack{12.\\ ..3}$ and 3 tableaux of $\substack{1.3\\ .2.}$. Suppose $x_2 = 2$ or $x_2 = 1$. Then we see by Table 1 that S must contain at least 2 tableaux of the form $\substack{12.\\ ..4}$ and 2 tableaux of the form $\substack{1.4\\ .2.}$. Since the tableaux corresponding to the above four forms are distinct, this implies $m \geq 10$, a contradiction. Thus, $x_2 = 0$. Similarly, either $x_4 = 0$ or $x_5 = 0$. Now $x_2 = 0$ implies $x_1 + x_4 \geq 0$. Since $x_1 = 1$, we must have $x_5 = 0$. Thus, we have $x_1 = 1$, $x_3 = x_4 = 4$, and $x_2 = x_5 = x_6 = 0$.

We will prove that this cannot happen. The proof will be independent of above assumptions and will be referred to later in the proof of this theorem.

Since $x_1 = 1$, we will assume S contains the tableau $T = \begin{matrix}123rs\\ xyz\end{matrix}$. Let T_{k_m} and T_j be as in (4.2), with b=2 and c=3. Then T in S adds -6 to the left hand side of (4.2). Thus, by Table 1 we need at least 3 tableaux in S of the form $\begin{matrix}12.\\ ..3\end{matrix}$. Since $x_2 = 0$, these tableaux have the form $\begin{matrix}125\\ ..3\end{matrix}$. Since $x_4 = 4$, we see by Table 1 that we must have exactly 3 tableaux of the form $\begin{matrix}125\\ ..3\end{matrix}$, otherwise (4.2) would not balance. Since the "charges" of the tableaux T_i in S must balance in each column of Table 1, one can show that the three tableaux in S of the form $\begin{matrix}125\\ ..3\end{matrix}$ must contain either {x,r,3}, {y,r,3}, and {z,r,3} or {x,s,3}, {y,s,3}, and {z,s,3} as bottoms. Without loss of generality, we will assume they contain {x,r,3}, {y,r,3}, and {z,r,3} as bottoms. Suppose the fourth tableau in S with {1,2,5} in the middle is $T' = \begin{matrix}125i\ell\\ uvw\end{matrix}$. We consider T_{k_m} and T_j as in (4.2) with b=2 and c=5, and with x,y,z,r, and s replaced by u,v,w,i, and ℓ, respectively. Then T' adds -6 to the left hand side of (4.2). Since $x_2 = 0$ and $5 \notin \{x,y,z\}$, S does not contain a tableau of the form $\begin{matrix}12.\\ ..5\end{matrix}$. Thus, we must balance (4.2) with the remaining three tableaux containing {1,2,5} in the middle. Since the "charges" of the tableaux in S must balance in each column of Table 1, one can show that these three tableaux must contain either {u,v,i}, {u,w,i}, and {v,w,i} or {u,v,ℓ}, {u,w,ℓ}, and {v,w,ℓ} as bottoms. This contradicts the fact that these three tableaux have {x,r,3}, {y,r,3}, and {z,r,3} as bottoms. Thus, we cannot have $x_1 = 1$, $x_3 = x_4 = 4$, and $x_2 = x_5 = x_6 = 0$. This implies $x_1 \neq 1$. Therefore, $x_1 = 0$.

Without loss of generality, $x_2 + x_3 = 4$, $x_4 + x_5 = 5$, and $x_4 \leq x_5$. If $x_4 = 0$, then $x_5 = 5$. Suppose T is the tableau $\begin{smallmatrix}153rs\\xyz\end{smallmatrix}$, and T$\varepsilon$S. Let T_{k_m} and T_j be as in (4.2), with b=5 and c=3. Since $x_4 = x_6 = 0$, S does not contain a tableau of the form $\begin{smallmatrix}15.\\..3\end{smallmatrix}$. Since each tableau in S with {1,5,3} in the middle adds -2, +2, or -6 to the left hand side of (3.2), we get that x_5 is even, a contradiction. Thus, $x_4 \neq 0$.

Suppose $x_4 = 1$. Then $x_5 = 4$ and $x_2 \geq 3$. This implies $0 \leq x_3 \leq 1$. If $x_3 = 0$, then $x_2 = 4$. In this case, $x_4 = 1$, $x_2 = x_5 = 4$, and $x_1 = x_3 = x_6 = 0$. If we apply the permutation (345) to all tableaux in S, we obtain an S' for which $x_1 = 1$, $x_3 = x_4 = 4$, and $x_2 = x_5 = x_6 = 0$, contradicting the fact that this cannot happen. Thus, $x_3 = 1$. We see by Table 1 that S must contain at least 3 tableaux of the form $\begin{smallmatrix}13.\\..4\end{smallmatrix}$ and 3 tableaux of the form $\begin{smallmatrix}1.4\\.3.\end{smallmatrix}$. Since

$x_1 = x_6 = 0$, these tableaux must have the forms $\substack{135\\ ..4}$ and $\substack{124\\ .3.}$. Since $x_4 = 1$, we see by Table 1 that S must contain at least 3 tableaux of the form $\substack{12.\\ ..5}$ and 3 tableaux of the form $\substack{1.5\\ .2.}$. Since $x_1 = x_6 = 0$, these tableaux must have the forms $\substack{124\\ ..5}$ and $\substack{135\\ .2.}$. Since $x_2 = 3$, we must have 3 tableaux in S of the form $\substack{124\\ .35}$. Since $x_5 = 4$, we must have at least 2 tableaux in S of the form $\substack{135\\ .24}$. But these 5 tableaux cannot be orthogonal ($f(3,1) = 3$). Thus, $x_4 \neq 1$. Therefore $x_4 = 2$. This implies $x_5 = 3$, $x_2 \geq 2$, and $1 \leq x_3 \leq 2$.

Suppose $x_3 = 1$. Then $x_2 = 3$. Since $x_3 = 1$, we see by Table 1 that S must contain at least 3 tableaux of the form $\substack{135\\ ..4}$ and 3 tableaux of the form $\substack{124\\ .3.}$. Since $x_4 = 2$, we see by Table 1 that S must contain at least 2 tableaux of the form $\substack{124\\ ..5}$ and 2 tableaux of the form $\substack{135\\ .2.}$. Since $x_2 = 3$, we must have at least 2 tableaux in S of the form $\substack{124\\ .35}$. Since $x_5 = 3$, we must have at least 2 tableaux in S of the form $\substack{135\\ .24}$. But these 4 tableaux cannot be orthogonal ($f(3,1) = 3$). Thus, $x_3 = 2$. This implies $x_2 = 2$. Since $x_2 = 2$, we see by Table 1 that S must contain at least 2 tableaux of the form $\substack{134\\ .2.}$. Since $x_3 = 2$, we see by Table 1 that S must contain at least 2 tableaux of the form $\substack{135\\ ..4}$ and 2 tableaux of the form $\substack{124\\ .3.}$. Since $x_4 = 2$, we see by Table 1 that S must contain at least 2 tableaux of the form $\substack{124\\ ..5}$ and 2 tableaux of the form $\substack{135\\ .2.}$. Since

$x_5 = 3$, we see by Table 1 that S must contain at least 1 tableau of the form $\begin{smallmatrix}1&3&4\\.&.&5\end{smallmatrix}$. We now put these results together. Since $x_2 = 2$, we must have 2 tableaux in S of the form $\begin{smallmatrix}1&2&4\\.&3&5\end{smallmatrix}$. Since $x_3 = 2$, we must have at least one tableau in S of the form $\begin{smallmatrix}1&3&4\\.&2&5\end{smallmatrix}$. Since $x_5 = 3$, we must have at least one tableau in S of the form $\begin{smallmatrix}1&3&5\\.&2&4\end{smallmatrix}$. But these 4 tableaux cannot be orthogonal $(f(3,1) = 3)$. Thus, no such S exists for (5,3).

We end this section by starting the classification of all complete sets of orthogonal tableaux for (4,3). By Theorem 2.11, f=14 for (4,3). Consider the frame (4,4). Suppose the first standard tableau T, occurs in a complete set of orthogonal tableaux S for (4,4). Then for each of the remaining 13 tableaux in S, at least one pair of numbers from {5,6,7,8} occurs in a column. Since there are 6 such pairs, one of the pairs, say {7,8}, occurs in a column of at least 3 tableaux in S. Up to similarity, we can put the 8 in the lower right hand corner of each tableau in S. If we remove the 8 from each tableau of S, we are left with a complete set of orthogonal tableaux for (4,3) in which 7 occurs in the end of at least three tableaux. Thus, any complete set of orthogonal tableaux for (4,3) is equivalent modulo (4,4) to a set in which the number 7 occurs in the end of at least three tableaux. If

Table 1. Summary of (4.2) if $l, b \notin \{r,s,x,y,z\}$

T_i \ T_k	1 crs xyz	2 csr xyz	3 rcs xyz	4 scr xyz	5 rsc xyz	6 src xyz	$\varepsilon_{ki}\varepsilon_{ij}$ [a]	TOTAL
	xrs	xrs	yrs	yrs	zrs	zrs	-	-2
lbc ...	xzr	xyr	yzr	xyr	yzr	xzr	+	+2
	xys	xzs	xys	yzs	xzs	yzs	+	+2
	xyz	xyz	xyz	xyz	xyz	xyz	-	-6
	rsc	rsc	rsc	rsc	rsc	rsc	-	-6
lb. ..c	zrc	yrc	zrc	xrc	yrc	xrc	+	+2
	ysc	zsc	xsc	zsc	xsc	ysc	+	+2
	yzc	yzc	xzc	xzc	xyc	xyc	-	-2

[a] T_j contains $\{l,b,c\}$ in the bottom.

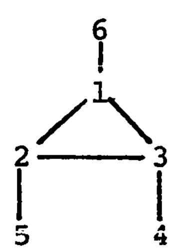
Table 2. Summary of (4.3) if $1 \notin \{t,x,y,z\}$

T_i \ T_k	bct xyz	cbt xyz	btc xyz	ctb xyz	tbc xyz	tcb xyz	$\varepsilon_{ki}\varepsilon_{ij}$ [a]	TOTAL[b]
1bc ...	xyt	xyt	xzt	xzt	yzt	yzt	+	+2
	xyz	xyz	xyz	xyz	xyz	xyz	−	−12
1b. ..c	xtc	ytc	xtc	ztc	ytc	ztc	+	+2
	xzc	yzc	xyc	yzc	xyc	xzc	−	−4
1.c .b.	ybt	xbt	zbt	xbt	zbt	ybt	+	+2
	ybz	xbz	ybz	xby	xbz	xby	−	−4
1.. .bc	tbc	tbc	tbc	tbc	tbc	tbc	+	+6
	zbc	zbc	ybc	ybc	xbc	xbc	−	−4

[a] T_j contains $\{1,b,c\}$ in the bottom.

[b] Total for both values of t.

S is a complete set of orthogonal tableaux for (4,3), let e be the number of tableaux in S in which 7 occurs in the end, and let b be the number of tableaux in S in which 7 occurs in the bottom. Without loss of generality, we will assume $e \geq 3$. Since the trace of a transposition in (4,3) is 4 [4, p. 148], we see by Theorem 3.10 that e = b-4. Thus, 7 occurs in the bottom of 7, 8, or 9 tableaux in S. We will consider the case when b=9. In this case, if we remove the 7 from each of these 9 tableaux, we obtain a complete set of orthogonal tableaux for (4,2). Up to similarity, there is only one such set for (4,2). Without loss of generality, we will use the set corresponding to



.

This set is (up to similarity):

1) 3256 14	2) 3156 24	3) 2346 15	4) 2146 35
5) 1345 26	6) 1245 36	7) 2316 45	8) 1325 46
9) 1234 . 56			

Note that before we add the 7 to row 2, column 3 of any tableau, we may want to interchange the two numbers in the end. After adding the 7 to these tableaux, we have four possibilities (up to similarity) for the resulting nine tableaux:

a) 1, 2, and 3 occur in the ends of tableaux 7, 8, and 9, respectively.

b) 1, 2, and 3 do not occur in the ends of tableaux 7, 8, and 9, respectively.

c) 1 and 2 occur in the ends of tableaux 7 and 8, but 3 does not occur in the end of tableau 9.

d) 1 occurs in the end of tableau 7, but 2 and 3 do not occur in the ends of tableaux 8 and 9.

We consider the cases individually.

a) In this case, we interchange 1 and 6 in tableau 7, 2 and 5 in tableau 8, and 3 and 4 in tableau 9. We then add the 7 to these three tableaux. This leaves {1,2,3}, {1,2,5}, {1,2,6}, {1,3,4}, {1,3,6}, {1,4,5}, {2,3,4}, {2,3,5}, {2,4,6}, and {3,5,6} as possible bottoms for the five tableaux containing 7 in the end. For the tableaux 1 and 2, 3 and 4, and 5 and 6, we must put the 7 under the same number for one of these pairs; otherwise, we would lose 6 more bottoms, leaving us with only 4. Without loss of generality, we add the 7 to tableaux 1 and 2 as they stand. This eliminates {1,2,5}, {1,4,5}, and {2,3,5} as possible bottoms. Thus, we must add the 7 to tableau 4 as it stands; otherwise, we lose 3 more bottoms. This eliminates {1,3,4} as a bottom. Thus, we must interchange 4 and 5 before adding the 7 to tableau 5. Also, we must add the 7 to tableaux 3 and 6 as they stand. This leaves {1,2,3}, {1,2,6}, {1,3,6}, {2,4,6}, and {3,5,6} as the bottoms for the remaining five tableaux. By Theorem 3.4, this determines the set (up to similarity).

b) In this case, we add the 7 to tableaux 7, 8, and 9 the way they stand. This leaves {1,2,4}, {1,3,5}, {1,4,6}, {1,5,6}, {2,3,6}, {2,4,5}, {2,5,6}, {3,4,5}, {3,4,6}, and {4,5,6} as possible bottoms for the five tableaux containing 7 in the end. For the tableaux 1 and 2, 3 and 4, and 5 and 6, we must put the 7 under the same number for one of these pairs; otherwise, we would lose 6 more bottoms. Without loss of generality, we add the 7 to tableaux 1 and 2 the way they stand. This eliminates {1,3,5}, {2,4,5}, and {3,4,5} as possible bottoms. Thus, we must switch 4 and 5 in tableau 6 before we add the 7. This eliminates {1,5,6} as a bottom. To make e = b-4 for the number 1, we must keep the remaining two bottoms which contain a 1. Thus, we must interchange the 4 and 6 in tableau 4 before we add the 7. This eliminates {2,5,6} as a bottom. Finally, we must add the 7 to tableau 3 the way it stands, and switch the 4 and 5 in tableau 5 before adding the 7. This leaves {1,2,4}, {1,4,6}, {2,3,6}, {3,4,6}, and {4,5,6} as bottoms for the remaining five tableaux.

c) In this case, we interchange 1 and 6 in tableau 7, and 2 and 5 in tableau 8. Then, adding the 7 to tableaux 7, 8, and 9 leaves {1,2,4}, {1,2,5}, {1,2,6}, {1,3,4}, {1,4,5}, {1,4,6}, {2,3,4}, {2,4,5}, and {2,4,6} as possible bottoms for the five tableaux containing 7 in the end. To make e = b-4 for the number 3, we must keep the remaining two bottoms which contain a 3. Thus, we must interchange 6 and 4 in tableaux 3 and 4, and 5 and 4 in tableaux 5 and 6, before we add the 7. This eliminates {1,2,5} and {1,2,6} as possible bottoms. If we were to add the 7 to tableaux 1 and 2 as they stand, we would be left with {1,2,4}, {1,3,4}, {1,4,6}, {2,3,4}, and {2,4,6} as bottoms for the remaining five tableaux. If we removed the 7 and the 4 from these five tableaux, we would be left with a set similar to a complete set of orthogonal tableaux for (3,2). But this set has the number 5 occurring five times in the top row. By section C, there is no complete set of orthogonal tableaux for (3,2) with this property. Thus, we cannot add the 7 to both tableaux 1 and 2 the way they stand. Similarly, we cannot interchange 5 and 6 in both tableaux 1 and 2 before adding the 7. Thus, we are left with two possibilities:

i) Add the 7 to tableau 1 the way it stands, and interchange 5 and 6 in tableau 2 before adding the 7. This leaves {2,4,5}, {1,2,4}, {1,4,6}, {1,3,4}, and {2,3,4} as bottoms for the remaining five tableaux.

ii) Add the 7 to tableau 2 as it stands, and interchange 5 and 6 in tableau 1 before adding the 7. This leaves {2,4,6}, {1,2,4}, {1,4,5}, {1,3,4}, and {2,3,4} as bottoms for the remaining five tableaux.

These two sets are not similar. In fact, these two sets are not equivalent modulo (4,4). To see this latter statement is true, add the 8 to all the tableaux, and note that the row structure of each set is different.

d) In this case, we interchange the 1 and 6 in tableau 7. Then adding the 7 to tableaux 7, 8, and 9 leaves {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}, {1,4,6}, {1,5,6}, {2,4,5}, and {3,4,5} as possible bottoms for the five tableaux containing 7 in the end. To make e = b-4 for the numbers 2 and 3, we must keep two bottoms containing the number 2 and two bottoms containing the number 3. Thus, we must interchange 5 and 6 in tableau 2, and 4 and 6 in tableau 4, before adding the 7. Suppose we were to add the 7 to tableaux 5 and 6 as they stand. Then, to make e = b-4 for the number 5, we would have to interchange 5 and 6 in tableau 1, before adding the 7. Also, we would have to interchange the 4 and 6 in tableau 3, before adding the 7. This would leave {1,2,5}, {1,3,5}, {1,4,5}, {2,4,5}, and {3,4,5} as bottoms for the remaining five tableaux. If we removed the 7 and the 5 from these five tableaux, we would be left with a set similar to a complete set of orthogonal tableaux for (3,2). But this set has the number 6 occurring five times in the top row. By section C, there is no complete set of orthogonal tableaux for (3,2) with this property. Thus, we cannot leave tableaux 5 and 6 as they stand. Similarly, we cannot interchange 4 and 5 in both tableaux 5 and 6. This leaves two possibilities:

i) Add the 7 to tableau 5 as it stands, and interchange 4 and 5 before adding the 7 to tableau 6.

ii) Add the 7 to tableau 6 as it stands, and interchange 4 and 5 before adding the 7 to tableau 5.

In both cases, we must interchange 5 and 6 in tableau 1, and 4 and 6 in tableau 3, before adding the 7. In case i), we are left with {1,2,4}, {1,3,5}, {1,4,5}, {2,4,5}, and {3,4,5} as bottoms for the remaining five tableaux. In case ii), we are left with {1,2,5}, {1,3,4}, {1,4,5}, {2,4,5}, and {3,4,5} as bottoms for the remaining five tableaux. These two sets are not equivalent modulo (4,4).

It turns out that of the six sets constructed, we have that there are three distinct sets up to equivalence modulo (4,4). These three sets are listed in the Appendix.

E. Sets which Split

In this section, we give a method for constructing complete sets of orthogonal tableaux which works well for certain frames.

Suppose we have a complete set of orthogonal tableaux for the frame $(m) = (m_1, m_2, \ldots, m_r)$, $m_1 + m_2 + \ldots + m_r = n$, in which n occurs at the end of a row in every tableau. We may, up to similarity, assume that if n occurs in row i, column m_i, then $m_i > m_{i+1}$, or $i = r$. By Theorems 2.12 and 3.3, if we remove n from each tableau, we will be left with complete sets of orthogonal tableaux for every frame which is derivable from (m) by removal of one square.

<u>Definition</u>: We say a complete set of orthogonal tableaux for the frame (m) <u>splits</u> if removal of n from each tableau yields a complete set of orthogonal tableaux for every frame which is derivable from (m) by removal of one square.

Often we will say a complete set of orthogonal tableaux splits if it is similar to a set which splits.

We now consider this process in reverse. It is sometimes possible to choose an appropriate complete set of orthogonal tableaux for each frame derivable from (m), such that, the addition of n to each tableau (to make it a tableau for (m)) will result in a complete set of orthogonal tableaux for (m).

As a simple example, we consider the frame (3,2). $\begin{array}{lll}123 & 124 & 143\\ 4 & 3 & 2\end{array}$ and $\begin{array}{ll}12 & 24\\ 34 & 13\end{array}$ are complete sets of orthogonal tableaux for the frames (3,1) and (2,2), respectively. If we add the 5 to each tableau, we obtain:

$$\begin{array}{lllll}123 & 124 & 143 & 125 & 245\\ 45 & 35 & 25 & 34 & 13\end{array} .$$

This is a complete set of orthogonal tableaux for (3,2).

Other examples of complete sets of orthogonal tableaux derived using this method appear in the Appendix.

V. CONCLUDING REMARKS

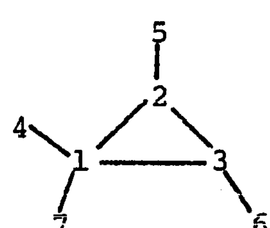
It is apparent from the preceding chapters that much work remains to be done on the problem of finding complete sets of orthogonal tableaux. One might hope that a classification similar to that for the frames (n-2,2) could be given for other types of frames, but such a classification does not, as yet, exist. Also, we have the fact that for some frames complete sets of orthogonal tableaux do not exist. In this chapter, we state some conjectures and suggest the possibility of using "in-complete" sets of orthogonal tableaux for frames for which complete sets of orthogonal tableaux do not exist. We also state some results whose proofs we have decided to omit.

We begin with the following conjecture:

For a given frame, if a complete set of orthogonal tableaux exists, then there exists a complete set of orthogonal tableaux that splits.

This does not say that every complete set of orthogonal tableaux is similar to a set that splits. In (3,2), there is one set (up to similarity) that does not split, and in (4,3), there are several such sets. However, the set in (3,2) is equivalent modulo (3,3) to a set that splits, and those in (4,3) are equivalent modulo (4,4) to sets that split. If this conjecture were true, then for a given frame, the method of Section E of Chapter IV will always produce a complete set of orthogonal tableaux if such a set exists. This conjecture would also imply that any frame containing a (5,2) or (2,2,1,1,1) has no complete set of orthogonal tableaux.

If, for some frame, no complete set of orthogonal tableaux exists, then we still have the possibility of choosing a maximal set of orthogonal tableaux for this frame. If the maximal set contains k tableaux, we look for f-k more tableaux such that A(I) will have determinant +1 or -1 (see the end of Chapter II). Preferably, A(I) will be lower triangular and contain a very small number of nonzero off-diagonal entries. As an example, we know that no complete set of orthogonal tableaux exists for the frame (5,2). Using the methods of Section C of Chapter IV, we can find a set of 13 orthogonal tableaux:



13567 24	12567 34	21367 45	31257 46	32457 16
31457 26	32147 56	23467 15	21467 35	13456 27
12456 37	12346 57	13245 67		

We choose $\begin{matrix}12356\\47\end{matrix}$ as the fourteenth tableau. Note that $e_i e_{14} = 0$ if $i<14$. Also, $e_{14}e_i = 0$ if $1<i<14$. Thus, in this case, the only nonzero off-diagonal entry of A(I) is in row 14, column 1. One might hope that this would always be the case, i.e., that the nonzero off-diagonal entries of A(I) could be restricted to row f, column 1. If this were the case, A(I) would be trivial to invert, and the matrix multiplication $A(I)^{-1}A(\pi)$, $\pi\varepsilon S_n$, would be trivial to perform (this is equivalent to having to calculate only one chain). As of yet, however, it is not known whether or not this is always the case.

Finally, we state without proof some results about $S(T)$, $S^2(T)$, and flat tableaux.

1) If T is a tableau for (3,2) or (4,3), then given any $T'\not\in S^2(T)$, T and T' are contained in a complete set of orthogonal tableaux. This is not always the case for the frame (4,2). Also, for (3,2) and (4,3), there exist tableaux T_1, T_2, and T_3 such that $T_i\not\in S^2(T_j)$ if $i\neq j$, and

yet T_1, T_2, and T_3 cannot be together in a complete set of orthogonal tableaux.

2) If T is a flat tableau and $T' \varepsilon S(T)$, then there exists a horizontal permutation p for T and a vertical permutation q for T such that $e' = \mathrm{sgn}(q)qep$.

3) Suppose S is a complete set of orthogonal tableaux for (6,3). We can show that if a pair of numbers occurs in the middle of some tableau in S, then it occurs in the middle of 7 tableaux in S. If we could show that it must occur in the middle of 8 tableaux in S, then we could prove that no such S exists.

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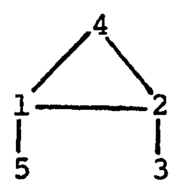
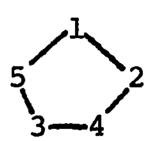
VIII. APPENDIX

We will list some examples of complete sets of orthogonal tableaux.

For the frame (3,2), there are two complete sets of orthogonal tableaux up to similarity:

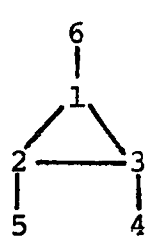


123	125	124	143	245
45	34	35	25	13



123	154	245	341	352
45	32	31	25	14 .

For the frame (4,2), there is one complete set of orthogonal tableaux up to similarity:



1234	1245	1246	1345	1356
56	36	53	26	42
1325	2346	2356	2316 .	
46	15	41	45	

For (4.3), we list the three distinct sets (up to equivalence modulo (4,4)) that were derived in Section D of Chapter IV:

2361 457	1352 467	1243 567	3256 147	3156 247	2146 357	1354 267
2346 157	1245 367	4567 231	3457 126	2547 136	1537 246	1427 356

2361 457	1352 467	1234 567	2364 157	2164 357	1354 267	1254 367
3256 147	3165 247	3617 254	6537 124	3527 164	6527 134	5617 234

2361 457	1352 467	1234 567	2364 157	2164 357	1354 267	1254 367
3265 147	3156 247	3517 264	5637 124	3627 154	5627 134	6517 234

There are many distinct complete sets of orthogonal tableaux for (3,2,1). Here we will list just one:

123 45 6	124 35 6	125 43 6	134 25 6	235 14 6	146 25 3	136 25 4
146 23 5	126 35 4	216 34 5	245 16 3	235 16 4	243 16 5	
215 46 3	213 46 5	214 36 5				

The following is a complete set of orthogonal tableaux for (3,3,1):

123 457 6	245 137 6	126 357 4	214 367 5	312 465 7	125 347 6
146 257 3	216 347 5	213 467 5	163 245 7	124 357 6	136 257 4
245 167 3	243 167 5	164 235 7	143 257 6	146 237 5	235 167 4
162 345 7	215 367 4	251 346 7			